On the foundations of nonlinear generalized functions I

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ABSTRACT. We construct a diffeomorphism invariant (Colombeau-type) differential algebra canonically containing the space of distributions in the sense of L. Schwartz. Employing differential calculus in infinite dimensional (convenient) vector spaces, previous attempts in this direction are unified and completed. Several classification results are achieved and applications to nonlinear differential equations involving singularities are given.

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1 Introduction

In his celebrated impossibility result ([40]), L. Schwartz demonstrated that the space $\mathcal{D}'(\Omega)$ of distributions over some open subset Ω of $\mathbb{R}^n$ cannot be embedded into an associative commutative algebra $(\mathcal{A}(\Omega), +, \circ)$ satisfying

(i) $\mathcal{D}'(\Omega)$ is linearly embedded into $\mathcal{A}(\Omega)$ and $f(x) \equiv 1$ is the unity in $\mathcal{A}(\Omega)$.

(ii) There exist derivation operators $\partial_i : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ $(i = 1, \ldots, n)$ that are linear and satisfy the Leibniz rule.

(iii) $\partial_i|_{\mathcal{D}'(\Omega)}$ is the usual partial derivative $(i = 1, \ldots, n)$.

(iv) $\circ|_{\mathcal{C}(\Omega)\times\mathcal{C}(\Omega)}$ coincides with the pointwise product of functions.

Since this result remains valid upon replacing $\mathcal{C}(\Omega)$ by $\mathcal{C}^k(\Omega)$ for any finite k, the best possible result would consist in constructing an embedding of $\mathcal{D}'(\Omega)$ as above with (iv) replaced by

(iv') $\circ|_{\mathcal{C}^\infty(\Omega)\times\mathcal{C}^\infty(\Omega)}$ coincides with the pointwise product of functions.

The actual construction of differential algebras satisfying these optimal properties is due to J. F. Colombeau ([9], [10], [11], [12]). The need for algebras of this type

arises, for example, from the necessity of considering non-linear PDEs where either the respective coefficients, the data or the prospective solutions are non-smooth. Classical linear distribution theory clearly does not permit the treatment of such problems. Colombeau algebras, on the other hand, have proven to be a useful tool for analyzing such questions (for applications in nonlinear PDEs, cf. e.g., [4], [5], [16], [14], [15], [29], [35], for applications to numerics, see e.g., [3], [6], [7], for applications in mathematical physics, e.g., [44], [32], [24] as well as the literature cited in these works). For alternative approaches to algebras of generalized functions, cf. [37], [38].

Since Colombeau's monograph [9], there have been introduced a considerable number of variants of Colombeau algebras, many of them adapted to special purposes. From the beginning, however, the question of the functor property of the construction was at hand as a crucial one: If $\mu : \tilde{\Omega} \to \Omega$ denotes a diffeomorphism between open subsets $\tilde{\Omega}, \Omega$ of $\mathbb{R}^s$, is it possible to extend the operation $\mu^* : f \mapsto f \circ \mu$ on smooth distributions on Ω to an operation $\hat{\mu}$ on the Colombeau algebra such that $(\mu \circ \nu)\hat{} = \hat{\nu} \circ \hat{\mu}$ and $(\mathrm{id})\hat{} = \mathrm{id}$ are satisfied? To phrase it differently, is it possible to achieve a diffeomorphism invariant construction of Colombeau algebras? As long as this question could not be answered in the positive, there remained the serious objection that there is no way of defining such algebras on manifolds, based on intrinsic terms, exclusively. (This discussion does not take into account the so called "special" or "simplified" variant of Colombeau's algebra whose elements are classes of nets of smooth functions indexed by $\varepsilon > 0$ (cf. [35], p. 109). Although diffeomorphism-invariant, these algebras lack a canonical embedding of distributions ([17], [41]), so we do not consider them here.)

The first variants of Colombeau algebras, though serving as a valuable tool in the treatment of non-linear problems, indeed did not have the property of diffeomorphism invariance: Some of the key ingredients used in defining them (in particular, the "test objects" (see section 3) being employed as well as the definition of the subsets $\mathcal{A}_q(\mathbb{R}^s)$ of the set of all test objects) turned out not to be invariant under the natural action of a diffeomorphism.

Colombeau and Meril in their paper [13] made the first decisive steps to remove this flaw by proposing a construction of Colombeau algebras which they claimed to be diffeomorphism invariant. As an essential tool, they had to use calculus on locally convex spaces. However, they did not give the details of the application of that calculus; moreover, their definition of the objects constituting the Colombeau algebra was not unambiguous and, which amounts to the most serious objection, their notion of test objects still was not preserved under the action of the diffeomorphism. Nevertheless, despite these defects (which, apparently, went unnoticed by nearly all workers in the field) their construction was quoted and used many times (see, e.g.,

[31], [18], [1], [45], [42], [34], [43], [19], [44], [2], [32]). It was only in 1998 that J. Jelínek in [28] pointed out the error in [13] by giving a (rather simple) counterexample. In the same paper, he presented another version of the theory avoiding the shortcomings of [13] and forming the basis for the approach taken here.

The present article is the first in a series of two papers. It is organized as follows: After fixing notation and terminology in section 2, a general scheme of construction for diffeomorphism-invariant Colombeau-type algebras of generalized functions is introduced in section 3. Section 4 gives a quick overview of calculus in convenient vector spaces providing the necessary results for the development of the theory. Especially with a view to applications (in particular: partial differential equations) we feel that this approach has several advantages over the concept of Silva-differentiability employed so far. Section 5 introduces a translation formalism that allows to freely switch between what we call the C- and J- (*Colombeau-* and *Jelinek-*) formalism of the diffeomorphism invariant theory to be constructed in section 7. In the actual construction of this algebra, smooth functions defined on sets denoted by $U_\varepsilon(\Omega)$ play a central rôle. Differentials of such functions are of utmost importance in the development of the theory. However, $U_\varepsilon(\Omega)$ is not a linear space. Sections 6 thus provides the framework necessary for doing calculus on $U_\varepsilon(\Omega)$. A complete presentation of the resulting diffeomorphism invariant algebra, based on the general construction scheme of section 3, is the focus of section 7. The sheaf-theoretic properties of this algebra are discussed in section 8. This is followed by a short section on the separation of testing procedures and definition of objects in algebras of generalized functions. Section 10 provides several new characterizations of the fundamental building blocks $\mathcal{E}_M$ and $\mathcal{N}$ of the algebra. In particular, these characterizations will constitute the key ingredient in obtaining an intrinsic description of the theory on manifolds ([26]). Finally, we present some applications to partial differential equations in section 11.

The second paper of this series gives a comprehensive analysis of algebras of Colombeau-type generalized functions in the range between the diffeomorphism-invariant quotient algebra $\mathcal{G}^d = \mathcal{E}_M/\mathcal{N}$ introduced in section 7 and (the smooth version of) Colombeau's original algebra $\mathcal{G}^e$ introduced in [10] (which, to be sure, is the standard version among those being independent of the choice of a particular approximation of the delta distribution). Three main results are established: First, a simple criterion describing membership in $\mathcal{N}$ (applicable to all types of Colombeau algebras) is given (section 13). Second, two counterexamples demonstrate that $\mathcal{G}^d$ is not injectively included in $\mathcal{G}^e$ (section 15); their construction is based on a completeness theorem for spaces of smooth functions in the sense of sections 4 and 6 (section 14). Finally, it is shown that in the range "between" $\mathcal{G}^d$ and $\mathcal{G}^e$ only one more construction leads to a diffeomorphism invariant algebra. In analyzing the latter, several classification results essential for obtaining an intrinsic description of $\mathcal{G}^d$ on

manifolds are derived (sections 16, 17). The concluding section 18 points out that also weaker invariance properties than with respect to all diffeomorphisms should be envisaged for Colombeau algebras, in particular regarding applications.

2 Notation and Terminology

Throughout this paper, Ω, $\tilde{\Omega}$ will denote non-empty open subsets of $\mathbb{R}^s$. For any $A \subseteq \mathbb{R}^s$, A° denotes its interior. $\mathcal{C}^\infty(\Omega)$ is the space of smooth, complex valued functions on Ω. If $f \in \mathcal{C}^\infty(\Omega)$ then Df denotes its (total) derivative. Also, we set $\check{f}(x) = f(-x)$. On any cartesian product, $\mathrm{pr_i}$ denotes the projection onto the i-th factor. For $r \in \mathbb{R}$, $[r]$ is the largest integer $\leq r$. We set $I = (0, 1]$. Concerning locally convex spaces our basic reference is [39]. In particular, by a locally convex space we mean a vector space endowed with a locally convex Hausdorff topology. The space of test functions (i.e., compactly supported smooth functions) on Ω is denoted by $\mathcal{D}(\Omega)$ and is equipped with its natural (LF)-topology; its dual, the space of distributions on Ω is termed $\mathcal{D}'(\Omega)$. The action of any $u \in \mathcal{D}'(\Omega)$ on a test function φ will be written as $\langle u, \varphi\rangle$. δ denotes the Dirac delta distribution. $K \subset\subset A$ ($A \subseteq \mathbb{R}^s$) means that K is a compact subset of A°. For $K \subset\subset \Omega$, $\mathcal{D}_K(\Omega)$ is the space of smooth functions on Ω supported in K. We set

$$\mathcal{A}_0(\Omega) = \{\varphi \in \mathcal{D}(\Omega) | \int \varphi(\xi)\, d\xi = 1\}$$

$$\mathcal{A}_q(\Omega) = \{\varphi \in \mathcal{A}_0(\Omega) | \int \xi^\alpha \varphi(\xi)\, d\xi = 0,\ 1 \leq |\alpha| \leq q,\ \alpha \in \mathbb{N}_0^s\} \qquad (q \in \mathbb{N})$$

$\mathcal{A}_{q0}(\Omega)$ is the linear subspace of $\mathcal{D}(\Omega)$ parallel to the affine space $\mathcal{A}_q(\Omega)$ ($q \in \mathbb{N}_0$). For any maps f, g, h such that $g \circ f$ and $f \circ h$ are defined we set $g_*(f) := g \circ f$ and $h^*(f) := f \circ h$. ∂_i resp. ∂^α always stand for $\frac{\partial}{\partial x_i}$ resp. $\frac{\partial^{|\alpha|}}{\partial x_i^\alpha}$ ($\alpha \in \mathbb{N}_0^s$).

For any locally convex space F the space $\mathcal{C}^\infty(\Omega, F)$ of smooth functions from Ω into F will always carry the topology of uniform convergence in all derivatives on compact subsets of Ω. In particular, a subset $\mathcal{B}$ of this space will be said to be bounded if, for any $K \subset\subset \Omega$ and any $\alpha \in \mathbb{N}_0^s$, the set $\{\partial^\alpha \phi(x) \mid \phi \in \mathcal{B}, x \in K\}$ is bounded in F. Observe that in case that the image of a map $\phi \in \mathcal{C}^\infty(\Omega, F)$ is contained in some affine subspace F_0 of F then the derivatives of ϕ take their values in the linear subspace parallel to F_0. For locally convex spaces E, F the space $\mathcal{C}^\infty(E, F)$ (resp. $\mathcal{C}^\infty(E)$ for $F = \mathbb{C}$) is introduced in section 4.

In what follows, $\mathcal{A}_0(\mathbb{R}^s)$ may be replaced by any closed affine subspace of $\mathcal{D}(\mathbb{R}^s)$. By $\mathcal{C}_b^{[\infty,\Omega]}(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$ we denote the space of all maps $\phi : I \times \Omega \to \mathcal{A}_0(\mathbb{R}^s)$ which are smooth with respect to the second argument and bounded in the sense that the

corresponding map $\hat{\phi} : I \to \mathcal{C}^\infty(\Omega, \mathcal{A}_0(\mathbb{R}^s))$ has a bounded image as defined above, i.e., for every $K \subset\subset \Omega$ and any $\alpha \in \mathbb{N}_0^s$, the set $\{\partial^\alpha(\hat{\phi}(\varepsilon))(x) \mid \varepsilon \in I, x \in K\}$ is bounded in $\mathcal{A}_0(\mathbb{R}^s)$ resp. $\mathcal{D}(\mathbb{R}^s)$, which, in turn, is equivalent to saying that

1. for every K as above and any $\alpha \in \mathbb{N}_0^s$, the supports of all $\partial_x^\alpha \phi(\varepsilon, x)$ ($\varepsilon \in I$, $x \in K$) are contained in some fixed bounded set (depending only on K) and
2. $\sup\{|\partial_\xi^\beta(\partial_x^\alpha(\hat{\phi}(\varepsilon))(x))(\xi)| \varepsilon \in I,\ x \in K,\ \xi \in \mathbb{R}^s\}$ (or, expressed in terms of ϕ itself) $\sup\{|\partial_\xi^\beta \partial_x^\alpha(\phi(\varepsilon, x))(\xi)| \varepsilon \in I,\ x \in K,\ \xi \in \mathbb{R}^s\}$ is finite.

$\mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$ is the subspace of $\mathcal{C}_b^{[\infty,\Omega]}(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$ whose elements are smooth in both arguments. Finally, for any $K \subset\subset \Omega$ and any $q \geq 1$ an element ϕ of $\mathcal{C}_b^\infty(I \times \Omega, \mathcal{D}(\mathbb{R}^s))$ is said to have asymptotically vanishing moments of order q on K if

$$\sup_{x \in K} \left| \int \xi^\alpha \phi(\varepsilon, x)(\xi)\, d\xi \right| = O(\varepsilon^q) \quad (1 \leq |\alpha| \leq q)\,.$$

For this notion to make sense it is obviously sufficient for ϕ to be defined on $(0, \varepsilon_0] \times K$ for some $\varepsilon_0 > 0$.

3 Scheme of construction

As was already pointed out in section 1, due to the lack of a canonical embedding of the space of distributions into "special" variants of Colombeau algebras we shall not consider these. Instead, we focus on "full" algebras (in the sense of [31], p. 31), distinguished by the fact that such a canonical embedding is always available. Elements of full Colombeau algebras are equivalence classes of functions R taking as arguments certain pairs (φ, x) consisting of a suitable test function $\varphi \in \mathcal{D}(\mathbb{R}^s)$ and a point x of Ω.

Every (full) Colombeau algebra is constructed according to the following blueprint (where **(Di)**, **(Tj)** stand for Definition i and Theorem j, respectively). **(D5)** and **(T6)**–**(T8)** are only relevant if a diffeomorphism invariant type of algebra is to be obtained.

(D1) $\mathcal{E}(\Omega)$ (the "basic space", see the remarks below); maps $\sigma : \mathcal{C}^\infty(\Omega) \to \mathcal{E}(\Omega)$, $\iota : \mathcal{D}'(\Omega) \to \mathcal{E}(\Omega)$.

(D2) Derivations D_i on $\mathcal{E}(\Omega)$ ($i = 1, \ldots, s$) extending the operators $\frac{\partial}{\partial x_i}$ of partial differentiation on $\mathcal{D}'(\Omega)$ resp. on $\mathcal{C}^\infty(\Omega)$, i.e., $D_i \circ \iota = \iota \circ \frac{\partial}{\partial x_i}$ and $D_i \circ \sigma = \sigma \circ \frac{\partial}{\partial x_i}$.

(D3) $\mathcal{E}_M(\Omega)$ ($\subseteq \mathcal{E}(\Omega)$; the subspace of "moderate" functions).

(D4) $\mathcal{N}(\Omega)$ ($\subseteq \mathcal{E}(\Omega)$; the subspace of "negligible" functions).

(T1) $\iota(\mathcal{D}'(\Omega)) \subseteq \mathcal{E}_M(\Omega)$, $\quad \sigma(\mathcal{C}^\infty(\Omega)) \subseteq \mathcal{E}_M(\Omega)$, $\quad (\iota - \sigma)(\mathcal{C}^\infty(\Omega)) \subseteq \mathcal{N}(\Omega)$;
$\iota(\mathcal{D}'(\Omega)) \cap \mathcal{N}(\Omega) = \{0\}$.

(T2) $\mathcal{E}_M(\Omega)$ is a subalgebra of $\mathcal{E}(\Omega)$.

(T3) $\mathcal{N}(\Omega)$ is an ideal in $\mathcal{E}_M(\Omega)$.

(T4) $\mathcal{E}_M(\Omega)$ is invariant under each D_i.

(T5) $\mathcal{N}(\Omega)$ is invariant under each D_i.

(D5) For each diffeomorphism $\mu : \tilde{\Omega} \to \Omega$, a map $\bar{\mu} : D_{\tilde{\Omega}} \to D_\Omega$ [1] is defined in a functorial way such that its "transpose" $\hat{\mu} : \mathcal{E}(\Omega) \to \mathcal{E}(\tilde{\Omega})$, $\hat{\mu}(R) := R \circ \bar{\mu}$, extends the usual effect μ has on distributions, i.e., $\hat{\mu} \circ \iota = \iota \circ \mu^*$ where for $u \in \mathcal{D}'(\Omega)$, $\mu^* u$ is defined by $\langle \mu^* u, \varphi \rangle := \langle u, (\varphi \circ \mu^{-1}) \cdot |\det D\mu^{-1}| \rangle$. Similarly, we require $\hat{\mu} \circ \sigma = \sigma \circ \mu^*$ on $\mathcal{C}^\infty(\Omega)$.

(T6) The class of "scaled test objects " (see below) is invariant under the action induced by μ.

(T7) $\mathcal{E}_M$ is invariant under $\hat{\mu}$, i.e., $\hat{\mu}$ maps $\mathcal{E}_M(\Omega)$ into $\mathcal{E}_M(\tilde{\Omega})$.

(T8) $\mathcal{N}$ is invariant under $\hat{\mu}$, i.e., $\hat{\mu}$ maps $\mathcal{N}(\Omega)$ into $\mathcal{N}(\tilde{\Omega})$.

(D6) $\mathcal{G}(\Omega) := \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$.

For $R \in \mathcal{E}_M(\Omega)$, the class $R + \mathcal{N}(\Omega)$ of R in $\mathcal{G}(\Omega)$ will be denoted by $[R]$.

The following comments are intended to motivate and clarify the preceding—admittedly very formal—definition schemes and theorems.

ad (D1): Here, $\mathcal{E}(\Omega)$ denotes some algebra of complex-valued functions having appropriate smoothness properties on a suitable domain $D_\Omega \subseteq \mathcal{D}(\mathbb{R}^s) \times \Omega$. σ has to be an injective algebra homomorphism, whereas ι just has to be linear and injective.

ad (D3), (D4): Membership of $R \in \mathcal{E}(\Omega)$ in $\mathcal{N}(\Omega)$ respectively $\mathcal{E}_M(\Omega)$ depends on the "asymptotic" behaviour of R on certain paths in $\mathcal{D}(\mathbb{R}^s) \times \Omega$, where the second component is constant whereas the first component, depending on ε as parameter, tends to the delta distribution weakly as $\varepsilon \to 0$. Essentially, these paths are obtained by applying the scaling operator $S_\varepsilon : \varphi \mapsto \frac{1}{\varepsilon^s}\varphi(\frac{\cdot}{\varepsilon})$ (thereby introducing the parameter ε) to so-called test objects. Typically, a test object is some fixed element $\varphi \in \mathcal{D}(\mathbb{R}^s)$ satisfying $\int \varphi = 1$ or a suitable bounded family $\phi(\varepsilon, x) \in \mathcal{D}(\Omega)$, parametrized by $\varepsilon \in I$, $x \in \Omega$, where again $\int \phi(\varepsilon, x)(\xi)\, d\xi \equiv 1$. Roughly speaking, R is defined to be

[1] Concerning $D_{\tilde{\Omega}}, D_\Omega$, see the remark on **(D1)** below.

negligible if the values R attains on those "scaled test objects" tend to zero faster than any positive power of ε, while it is called moderate if these values are bounded by some fixed (negative) power of ε. In both cases, convergence in each derivative, uniformly on compact subsets of Ω, is required. We will refer to those defining procedures as **testing for negligibility** resp. **moderateness** (see also section 9).

ad (T1), (T3): $\mathcal{N}(\Omega)$ has to be large enough to contain all $\sigma(f) - \iota(f)$ ($f \in \mathcal{C}^\infty(\Omega)$) (this renders $\iota\,|_{\mathcal{C}^\infty(\Omega)}$ an algebra homomorphism by passing to a quotient by $\mathcal{N}(\Omega)$), however small enough to intersect $\mathcal{D}'(\Omega)$ just in $\{0\}$ (this guarantees $\mathcal{D}'(\Omega)$ to be contained injectively in the quotient by $\mathcal{N}(\Omega)$). $\mathcal{E}_M(\Omega)$, on the other hand, clearly has to be large enough to contain $\mathcal{C}^\infty(\Omega)$ and $\mathcal{D}'(\Omega)$ (via σ resp. ι), yet small enough such that $\mathcal{N}(\Omega)$ is an ideal in it: This will allow us to form the quotient $\mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$.

ad (D5): μ^* as defined above extends $\mu^* : f \mapsto f \circ \mu$ where the latter is viewed as the action induced by μ on the smooth distribution $f \in \mathcal{C}^\infty(\Omega)$. Hence we regard distributions (and, in the sequel, non-linear generalized functions) as generalizations of *functions* on the respective open set, acting as functionals on (smooth, compactly supported) densities. This is in agreement with, for example, [27], however has to be distinguished clearly from constructing distributions as distributional *densities*, acting on (smooth, compactly supported) functions, as it is done, e.g., in [20].

ad (T7), (T8): Because of the forms of **(D3)** and **(D4)** as tests to be performed on the elements R of $\mathcal{E}(\Omega)$, with the appropriate type of (scaled) test objects being inserted, **(T7)** as well as **(T8)** follow immediately from **(T6)**, taking into account **(D5)**.

ad (D6): By this definition, $\mathcal{G}(\Omega)$ is a differential algebra containing $\mathcal{D}'(\Omega)$ via ι followed by the canonical quotient map (**(T1)**–**(T5)**); by abuse of notation, we will denote this embedding also by ι. Each diffeomorphism $\mu : \tilde{\Omega} \to \Omega$ induces a map $\hat{\mu} : \mathcal{G}(\Omega) \to \mathcal{G}(\tilde{\Omega})$ extending the usual action of μ on distributions such that composition and identities are preserved by $\mu \mapsto \hat{\mu}$ (**(T7)**, **(T8)**).

Without the requirement of diffeomorphism invariance (as, for example, in [9]), the smoothness property of R mentioned above only needs to refer to the variable x in the pair (φ, x), thus involving only classical calculus. However, as mentioned already in the introduction, to obtain a diffeomorphism invariant algebra we also have to consider smoothness with respect to the test function φ. Therefore, in the following section, we are going to outline the elements of calculus on locally convex spaces which are required for the subsequent constructions. The path we will pursue in this respect is different from the approaches taken so far and, in our view, has some decisive advantages over these.

4 Calculus

In the first versions of Colombeau algebras (on $\mathbb{R}^n$ or open subsets thereof), the main ingredient was the algebra of smooth functions $\varphi \mapsto R(\varphi)$ on the ((LF)-)space $\mathcal{D}$ of test functions (see [9]). Thus, from the very beginning, there had to be a theory of differentiation on (certain non-Banach) locally convex spaces at the basis of the construction of these algebras.

Colombeau's approach in [9] employs the notion of *Silva-differentiability* ([46], [8]) where a map $f : E \supseteq U \to F$ from an open subset U of a locally convex space E into another locally convex space F is called Silva-differentiable in $x \in U$ if there exists a bounded linear map (called $f'(x)$) $E \to F$ such that the restriction of the corresponding remainder function to sufficiently small homothetic images of bounded subsets may be viewed as a map between suitable normed spaces and satisfies a condition thereon which is completely analogous to the classical remainder condition for Fréchet-differentiable maps.

In later versions, Colombeau managed to circumvent this necessity by introducing an additional variable $x \in \mathbb{R}^n$ into R which could carry the burden of smoothness: For the construction of the algebra $\mathcal{G}(\Omega)$ of [10] he now used functions $R(\varphi, x)$ which, for each fixed φ from (a certain affine subspace of) $\mathcal{D}$, are smooth in x (in the usual elementary sense—hence the title of [10]) whereas the dependence on φ is completely arbitrary; φ just plays the rôle of a parameter in this setting. Apart from simplifying the general setup of the theory the introduction of x as a separate variable was also crucial for solving differential equations in $\mathcal{G}(\Omega)$.

However, when Colombeau and Meril in [13] began to develop a diffeomorphism invariant version of the algebra $\mathcal{G}(\Omega)$ of [10], they had to reintroduce the smooth dependence of R on φ: Under the action of a diffeomorphism μ, φ changes to some $\tilde{\varphi}_x$ *depending on* x. For the smoothness of the μ-transform of R (which, according to **(D5)**, is of the form $(\hat{\mu}R)(\varphi, x) = R(\bar{\mu}(\varphi, x)) = R(\tilde{\varphi}_x, \mu x))$ with respect to x, obviously the smooth dependence of R also on its first argument φ is needed ([13], p. 263). Concerning calculus on locally convex spaces, the authors—as the first of them did already in [9]—refer to [8]. Omitting any details in this respect, they rather invite the reader to admit the respective smoothness properties (p. 263).

Jelínek in [28] includes a section on calculus (items 9–16): In addition to [8], he quotes [46] as reference for some results needed. The relevant statements are formulated in terms of higher Fréchet differentials.

Contrary to the above, we prefer to base our presentation on the notion of smoothness as it is outlined in [30]. This approach seems to us to have a number of striking advantages: On the one hand, the basic definition is very simple and easy to work

with, a smooth map between locally convex spaces E, F being one that takes smooth curves $\mathbb{R} \to E$ to smooth curves $\mathbb{R} \to F$ (by composition); the notion of a smooth curve into a locally convex space obviously is without problems. We will denote by $\mathcal{C}^\infty(E, F)$ the space of smooth maps between E and F. For $\mathcal{C}^\infty(E, \mathbb{C})$, we will simply write $\mathcal{C}^\infty(E)$. On the other hand, all the basic theorems of differential calculus can be reconstructed in this setting (see, e.g., the version of the mean value theorem given in 4.5) and more than that (see, e.g., the exponential laws stated in 4.2 and 4.3 below and the differentiable uniform boundedness principle 4.7). As smooth curves are continuous, the above definition of smoothness carries over to open subsets of locally convex spaces. We will make use of this in the sequel and want to note that in any of the theorems of this section, we may replace the respective locally convex (domain) spaces by open subsets thereof whenever their linear structure is not needed.

This notion of smoothness is a weaker one than Silva-differentiability but turns out to be equivalent for a huge class of spaces, e.g. those which are complete and Montel so that the two notions coincide in particular on the regular[2] strict inductive limit $\mathcal{D}(\Omega)$ of Fréchet spaces and each closed subspace thereof.

The seeming drawback of this (and any other reasonable such as Colombeau's above-mentioned) theory of differentiation is the fact that smooth maps (resp. their differentials) need no longer be continuous. The fundamental rôle played by continuity in the classical context is taken over by the notion of boundedness: Indeed, the difference quotients of smooth curves converge in a stronger sense than the topological one, so that continuity is not a necessary property for a map to be smooth. In order to be able to test smoothness by composition with suitable families of linear functionals (see, e.g., 4.7) one needs, in addition, a completeness property which is weaker than completeness of the locally convex topology. Separated bornological locally convex spaces which have this property are called *convenient spaces* and are in some sense the most general class of linear spaces in which one can perform differentiation and integration. As for each locally convex space there exists a finer bornological locally convex topology with the same bornology, i.e., the same system of bounded sets, bornologicity of the topology is not essential. It will be enough for our purpose to confine ourselves to the particular case of complete locally convex spaces.

In the sequel, we will endow the space $\mathcal{C}^\infty(\mathbb{R}, F)$ of smooth curves into the locally convex space F with the locally convex topology of uniform convergence on compact intervals in each derivative separately. More generally, we may consider on the space

[2] A strict inductive limit $\varinjlim E_\alpha$ is called regular if each bounded subset is contained in some E_α. Note that every strict inductive limit of an increasing sequence E_n is regular, as is $\mathcal{D}(M)$ for any paracompact (not necessarily separable) smooth manifold M.

$\mathcal{C}^\infty(E,F)$ the initial locally convex topology induced by the pullbacks along smooth curves $\mathbb{R} \to E$. It can be shown that the bounded sets associated with this topology are the same as the ones associated with the topology of uniform convergence on compact subsets in each differential (as defined for such maps in 4.4) separately. Moreover, as mentioned in the introduction of this paper, for complete F, the latter is again complete; see section 14 of the second part of this series ([25]) for details.

Testing of smoothness is particularly simple in the case of a linear map: A linear map is smooth if and only if it is bounded. $L(E,F)$ will stand for the space of bounded (smooth) linear maps between E, F.

4.1 Theorem. *A map f from $\mathcal{D}(\Omega)$ into a locally convex space E is smooth if and only if for each $K \subset\subset \Omega$, the restriction of f to $\mathcal{D}_K(\Omega)$ is smooth.*

Proof. For the non-trivial part of the proof, consider a smooth curve $c : \mathbb{R} \to \mathcal{D}(\Omega)$. Its restriction to any bounded interval J has a relatively compact, hence bounded image. Therefore, c maps J into some $\mathcal{D}_K(\Omega)$ and the same holds for each derivative of c since $\mathcal{D}_K(\Omega)$ is a closed subspace of $\mathcal{D}$. By assumption, $f \circ c$ is smooth on J. Since smoothness is a local property, we are done. □

The obvious generalization of the preceding theorem is true for any strict inductive limit of a sequence of Fréchet spaces. Its trivial part has an important consequence: $\mathcal{D}_K(\Omega)$ being a Fréchet space, the restriction to $\mathcal{D}_K(\Omega)$ of any smooth map f from $\mathcal{D}(\Omega)$ to any metrizable locally convex space E is continuous: Both on $\mathcal{D}_K(\Omega)$ and E the so-called c^∞-topology (see [30]) coincides with the metric topology ([30], 4.11.(1)); moreover, smooth maps are continuous with respect to the c^∞-topology ([30], p. 8).

One of the particular features of the Frölicher-Kriegl-theory which considerably simplify its application is the *exponential law* (cf. Theorem 3.12 and Corollary 3.13 in [30]):

4.2 Theorem. *Let E, F, G be locally convex spaces. Then the two spaces $\mathcal{C}^\infty(E \times F, G)$ and $\mathcal{C}^\infty(E, \mathcal{C}^\infty(F, G))$ are isomorphic algebraically and bornologically, i.e., they have the same bounded sets.*

Replacing $\mathcal{C}^\infty$ by L in 4.2 yields the exponential law for linear smooth maps. By iteration one obtains (see Proposition 5.2 in [23]):

4.3 Theorem. *Let $n, k \in \mathbb{N}$ and E_i, F $(i = 1, \dots, n+k)$ locally convex spaces. Then there is a bornological isomorphism*

$$L(E_1, \dots, E_{n+k}; F) \cong L(E_1, \dots, E_n; L(E_{n+1}, \dots, E_{n+k}; F)).$$

For later use, we present the analoga of items 10–16 in [28] in the setting of [30]:

4.4 Theorem. *(Theorem 3.18 and Corollary 5.11 in [30]) Let E, F be locally convex spaces. Then the differentiation operator* $\mathrm{d} : \mathcal{C}^\infty(E, F) \to \mathcal{C}^\infty(E, L(E, F))$ *given by*

$$\mathrm{d}f(x)v := \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t}$$

exists and is linear and bounded (smooth). Hence, for $n \in \mathbb{N}$ one can form the iterated differentiation operator

$$\mathrm{d}^n : \mathcal{C}^\infty(E, F) \to \mathcal{C}^\infty(E, L(E, \dots, L(E; F)\dots)) \cong \mathcal{C}^\infty(E, L(E, \dots, E; F))$$

which is smooth and linear and has values in $\mathcal{C}^\infty(E, L_{sym}(E, \dots, E; F))$, where $L_{sym}(E, \dots, E; F)$ stands for the space of smooth n-linear symmetric maps between $E \times \cdots \times E$ and F. Also, the chain rule holds:

$$\mathrm{d}(f \circ g)(x)v = \mathrm{d}f(g(x))\mathrm{d}g(x)v.$$

It is shown in [30], 1.4, that, given a curve which is smooth from (an open neighborhood of) $\mathbb{R} \supseteq [a, b]$ to E, the difference quotient $\frac{c(b)-c(a)}{b-a}$ is an element of $\overline{\mathrm{conv}}\{c'(t) : t \in [a, b]\}$, where $\overline{\mathrm{conv}}$ denotes the closed convex hull. By virtue of the chain rule given in 4.4, this is equivalent to

4.5 Proposition. *(Mean Value Theorem) Let $f : E \supseteq U \to F$ be smooth, where U is an open neighborhood of a segment $[x, x+v] \subseteq E$. Then*

$$f(x+v) - f(x) \in \overline{\mathrm{conv}}\{\mathrm{d}f(x+tv)(v) : t \in [0, 1]\}.$$

As a consequence of 4.2, for each smooth map $f \in \mathcal{C}^\infty(F, G)$, the maps $f_* : \mathcal{C}^\infty(E, F) \to \mathcal{C}^\infty(E, G)$ and $f^* : \mathcal{C}^\infty(G, E) \to \mathcal{C}^\infty(F, E)$ are smooth. In particular, for a smooth map $f \in \mathcal{C}^\infty(E \times F, G)$ we may define smooth linear "operators of partial differentials" $\mathrm{d}_1, \mathrm{d}_2$ as

$$\mathrm{d}_1 := (\iota_E^*)_* \circ \mathrm{d} : \mathcal{C}^\infty(E \times F, G) \to \mathcal{C}^\infty(E \times F, L(E, G))$$

and

$$\mathrm{d}_2 := (\iota_F^*)_* \circ \mathrm{d} : \mathcal{C}^\infty(E \times F, G) \to \mathcal{C}^\infty(E \times F, L(F, G)),$$

where ι_E, ι_F denote the natural embeddings of E resp. F into $E \times F$. Obviously, we have

$$\mathrm{d}_1 f(x)(v) = \mathrm{d}f(x)(\iota_E(v)) = \lim_{t\to 0} \frac{f(x + t\iota_E(v)) - f(x)}{t},$$

which yields an alternative definition of d_1, which makes sense also for maps $f : E \times F \to G$ which are not a priori known to be smooth on $E \times F$.

4.6 Proposition. *A map on $E \times F$ is smooth if and only if both partial differentials* $\mathrm{d}_1, \mathrm{d}_2$ *exist and are smooth as maps on $E \times F$. In this case the differential* d *equals the sum* $(\mathrm{pr}_1^*)_* \circ \mathrm{d}_1 + (\mathrm{pr}_2^*)_* \circ \mathrm{d}_2$ *of the partial differentials; the iterated mixed second derivatives coincide via the isomorphism $L(E, L(F, G)) \cong L(F, L(E, G))$ which is a consequence of 4.2.*

Proof. Necessity follows by what has been remarked above together with the symmetry of iterated derivatives stated in 4.4. For sufficiency, consider the map $\tilde{\mathrm{d}}f \in \mathcal{C}^\infty(E \times F, L(E \times F, G))$ defined by $\tilde{\mathrm{d}}f(x)(v_1, v_2) := \mathrm{d}_1 f(x)(v_1) + \mathrm{d}_2 f(x)(v_2)$. Then obviously for fixed x the map $(t, v) \mapsto \tilde{\mathrm{d}}f(x + tv)(v)$ is smooth from $[0, 1] \times E \times F \to G$ and hence can be viewed as an element of $\mathcal{C}^\infty([0, 1], \mathcal{C}^\infty(E \times F, G))$. By [30], 2.7, a smooth curve is Riemann integrable, the Riemann integral leads again into $\mathcal{C}^\infty(E \times F, G)$ and commutes with the application of smooth linear maps. It follows that the map

$$v \mapsto f(x) + \int_0^1 \tilde{\mathrm{d}}f(x + tv)(v)dt$$

is smooth on $E \times F$ and it suffices to verify that the expression on the right hand side equals $f(x+v)$ in order to obtain smoothness of f on $E \times F$. For this, note that for each fixed segment $[x, x + v]$, we can recover the claimed identity from the finite dimensional one by composing the restriction of f to the segment with bounded linear functionals. □

The *differentiable uniform boundedness principle* (see 4.4.7 in [23]) constitutes an extremely useful tool for testing smoothness of linear maps into spaces of smooth functions:

4.7 Theorem. *Let E, F, G be locally convex spaces, E, G complete. A linear map $E \to \mathcal{C}^\infty(F, G)$ is smooth if and and only if its composition with the evaluation* ev_x *for each $x \in F$ is smooth.*

If we endow the space $\mathcal{C}^\infty(X, \mathbb{R})$ (in the present paper, X will be one of the spaces $\mathcal{D}(\Omega), \mathcal{D}(\Omega) \times \Omega, \mathcal{A}_0(\Omega) \times \Omega$ or $\mathcal{A}_0(\mathbb{R}^s) \times \Omega$) with the topology of uniform convergence on compact subsets in each derivative, i.e., in each iterated differential separately, then by considering the corresponding seminorms one sees that taking the differential constitutes a continuous linear operation. To be precise, the space $\mathcal{A}_0(\mathbb{R}^s)$ is not a linear space itself but the affine image of the closed linear subspace $E := \mathcal{A}_{00}(\mathbb{R}^s) \subseteq \mathcal{D}(\mathbb{R}^s)$ and may be identified with the latter. A map on $\mathcal{A}_0(\mathbb{R}^s)$ is then said to be smooth if it is the pullback of a smooth map on E under the affine isomorphism. We say that the smooth structure on $\mathcal{A}_0(\mathbb{R}^s)$ is induced by its isomorphism with

E. This is a simple example of the notion of a *smooth space* as introduced in [23]. Locally convex spaces may be viewed as smooth spaces with a compatible linear structure.

4.8 Proposition. *The following maps (to be defined in section 5) are smooth: The linear maps $S_\varepsilon : \mathcal{D}(\mathbb{R}^s) \to \mathcal{D}(\mathbb{R}^s)$, $T_x : \mathcal{D}(\mathbb{R}^s) \to \mathcal{D}(\mathbb{R}^s)$ and $(\varphi, x) \mapsto \bar{\mu}^X(\varphi, x)$, $R \mapsto \hat{\mu}^X R$ ($X \in \{C, J\}$), as well as the non-linear maps S, T, $x \mapsto T_x$, $x \mapsto T_x\varphi$, $(\varphi, x) \mapsto T_x\varphi$.*

Proof. Smoothness of $S_\varepsilon, T_x, \bar{\mu}^X, \hat{\mu}^X$ follows by our remarks preceding 4.1 and following 4.5, respectively, as each of these maps is essentially a pullback of a smooth map by definition. As the map $S : (\varepsilon, \varphi) \mapsto S_\varepsilon\varphi)$ is linear in φ, it follows by the exponential law 4.2 and the uniform boundedness principle 4.7 that S is smooth iff it is separately smooth, i.e., if and only if the maps S_ε and $(\varepsilon \mapsto S_\varepsilon\varphi)$ are smooth. While smoothness of the former is already established, the latter is a curve which is obviously smooth off 0 and we are done. In a similar fashion, we obtain smoothness of T and all the maps associated with it. □

5 C- and J-formalism

Colombeau in [10] and in [13] (together with Meril) on the one hand and Jelínek in [28] on the other hand used different, yet equivalent formalisms to describe their respective constructions of Colombeau algebras: For embedding the space $\mathcal{D}'(\mathbb{R}^s)$ of distributions on $\mathbb{R}^s$ into the space $\mathcal{E}_M(\mathbb{R}^s)$ of representatives of generalized functions, they chose different (linear injective) maps which we denote by ι^C ([10], [13]) and ι^J ([28], compare also [9]), respectively. On a distribution given by a smooth function f on $\mathbb{R}^s$, ι^C and ι^J are defined by

$$(\iota^C f)(\varphi, x) := \int f(y)\varphi(y - x)\, dy \tag{1}$$

resp.

$$(\iota^J f)(\varphi, x) := \int f(y)\varphi(y)\, dy. \tag{2}$$

Here, φ denotes a test function from the subspace $\mathcal{A}_0(\mathbb{R}^s)$ of $\mathcal{D}(\mathbb{R}^s)$ while $x \in \mathbb{R}^s$. There are good reasons for either of these choices of the embedding; we are going to discuss their respective merits below. In this section we show that both formalisms are actually equivalent and establish a translation formalism allowing to change from one setting to the other at any stage of the presentation.

5.1 Definition. *For $\varepsilon \in I$ and $x \in \mathbb{R}^s$ define the following operators:*

$$
\begin{aligned}
T_x : \mathcal{D}(\mathbb{R}^s) \ni \varphi &\mapsto T_x\varphi := \varphi(.-x) \in \mathcal{D}(\mathbb{R}^s) &(3)\\
S_\varepsilon : \mathcal{D}(\mathbb{R}^s) \ni \varphi &\mapsto S_\varepsilon\varphi := \frac{1}{\varepsilon^s}\varphi\left(\frac{.}{\varepsilon}\right) \in \mathcal{D}(\mathbb{R}^s) &(4)\\
S : (0,\infty) \times \mathcal{D}(\mathbb{R}^s) \ni (\varepsilon,\varphi) &\mapsto S_\varepsilon\varphi \in \mathcal{D}(\mathbb{R}^s) &(5)\\
T : \mathcal{D}(\mathbb{R}^s) \times \mathbb{R}^s \ni (\varphi,x) &\mapsto T(\varphi,x) := (T_x\varphi, x) \in \mathcal{D}(\mathbb{R}^s) \times \mathbb{R}^s &(6)\\
S^{(\varepsilon)} : \mathcal{D}(\mathbb{R}^s) \times \mathbb{R}^s \ni (\varphi,x) &\mapsto S^{(\varepsilon)}(\varphi,x) := (S_\varepsilon\varphi, x) \in \mathcal{D}(\mathbb{R}^s) \times \mathbb{R}^s. &(7)
\end{aligned}
$$

T_x and S_ε are linear. All the operators introduced in the preceding definition are one-one and onto; moreover, they are continuous and smooth with respect to the natural topologies (see section 4).

In a next step, we take (1) and (2) as a starting point for the determination of suitable domains for representatives of generalized functions on an open subset Ω of $\mathbb{R}^s$: Assuming $x \in \Omega$ in (1) and (2), it is immediate that in (2) φ has to have its support in Ω, whereas for (1) to be well-defined for any smooth function f on Ω, the support of φ must be contained in $\Omega - x$. This motivates the introduction of the following sets:

5.2 Definition. *Let $\varepsilon \in I$.*

$$
\begin{aligned}
U(\Omega) &:= T^{-1}(\mathcal{A}_0(\Omega) \times \Omega) \qquad = \{(\varphi,x) \in \mathcal{A}_0(\mathbb{R}^s) \times \Omega \mid \operatorname{supp}\varphi \subseteq \Omega - x\}\\
U_\varepsilon(\Omega) &:= (S^{(\varepsilon)})^{-1}(U(\Omega)) =\\
&= (TS^{(\varepsilon)})^{-1}(\mathcal{A}_0(\Omega) \times \Omega) = \{(\varphi,x) \in \mathcal{A}_0(\mathbb{R}^s) \times \Omega \mid \operatorname{supp}\varphi \subseteq \varepsilon^{-1}(\Omega - x)\}
\end{aligned}
$$

The notation $U(\Omega)$ is due to Colombeau ([10], 1.2.1). By definition, the maps $T : U(\Omega) \to \mathcal{A}_0(\Omega) \times \Omega$ and $S^{(\varepsilon)} : U_\varepsilon(\Omega) \to U(\Omega)$ are algebraic isomorphisms in the sense that they are bijective and linear in the first argument. The question of topology, however, is somewhat subtle: Let τ_Ω denote the product of the (LF)-topology of $\mathcal{D}(\Omega)$ and the Euclidean topology on Ω; abbreviate $\tau_{\mathbb{R}^s}$ as τ_0. Then on $\mathcal{A}_0(\Omega) \times \Omega$, the topology τ_Ω without doubt is the appropriate one to consider, rather than (the restriction of) τ_0. For $U(\Omega)$, on the other hand, the topology τ_1 induced by τ_0 and the topology $\tau_2 := T^{-1}\tau_\Omega$ both seem to be natural choices. (Note that τ_1 can be obtained equally as $T^{-1}\tau_0$, due to T being a homeomorphism with respect to τ_0.) As the following example (which can easily be generalized to arbitrary non-trivial open subsets of $\mathbb{R}^s$) shows, τ_1 is strictly coarser than τ_2 in general.

5.3 Example. Let $\Omega := \{x \in \mathbb{R} \mid x > -1\}$. Choose $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}\varphi = [0,1]$ and $\int \varphi = 0$. Pick any $\rho \in \mathcal{A}_0(\Omega)$ such that $\operatorname{supp}\rho \subseteq [1,2]$. Letting $\psi_n :=$

$\rho + \frac{1}{n}\varphi(. + \frac{n-1}{n}) \in \mathcal{A}_0(\Omega)$, it is easy to check that $T^{-1}(\psi_n, 0) = (\psi_n, 0) \in U(\Omega)$ tends to $(\rho, 0) \in U(\Omega)$ with respect to τ_1, yet is not convergent (in fact, not even bounded) with respect to τ_2.

The situation is similar in the case of $U_\varepsilon(\Omega)$: Apart from the topology $\tau_{1,\varepsilon}$ induced by the topology τ_0 of $\mathcal{A}_0(\mathbb{R}^s) \times \mathbb{R}^s$ via inclusion, the natural topology τ_Ω of $\mathcal{A}_0(\Omega) \times \Omega$ via $TS^{(\varepsilon)}$ induces a topology $\tau_{2,\varepsilon}$ which, in general, is strictly finer than $\tau_{1,\varepsilon}$.

Now, in order to have the respective formalisms of Colombeau and Jelínek equivalent, we want $T : U(\Omega) \to \mathcal{A}_0(\Omega) \times \Omega$ and $S^{(\varepsilon)} : U_\varepsilon(\Omega) \to U(\Omega)$ to be also topological isomorphisms (hence diffeomorphisms). This amounts to endowing $U(\Omega)$ and $U_\varepsilon(\Omega)$ with the topologies τ_2 resp. $\tau_{2,\varepsilon}$ induced via T resp. $TS^{(\varepsilon)}$. Thus we adopt the following convention:

Whenever questions of topology (in particular, boundedness) or smoothness on $U(\Omega)$ or $U_\varepsilon(\Omega)$ are discussed, we consider their topologies to be τ_2 resp. $\tau_{2,\varepsilon}$, i.e., those induced by the natural topology of $\mathcal{A}_0(\Omega) \times \Omega$ via T resp. $TS^{(\varepsilon)}$.

To phrase it differently, $U(\Omega)$ can be viewed as (infinite-dimensional) smooth manifold, modelled over $\mathcal{A}_0(\Omega) \times \Omega$, having an atlas consisting of a single chart T. A similar statement is valid for $U_\varepsilon(\Omega)$ and $TS^{(\varepsilon)}$. The importance as well as the subtlety of distinguishing between τ_1 and τ_2 are highlighted in example 5.9 below.

We are now able to introduce the basic spaces of smooth functions on which the construction of diffeomorphism invariant Colombeau algebras is built.

5.4 Definition.

$$\begin{aligned} \mathcal{E}^J(\Omega) &:= \mathcal{C}^\infty(\mathcal{A}_0(\Omega) \times \Omega) && (8) \\ \mathcal{E}^C(\Omega) &:= \mathcal{C}^\infty(U(\Omega)) && (9) \end{aligned}$$

By our above choice of topologies, T^* indeed maps $\mathcal{E}^J(\Omega)$ bijectively onto $\mathcal{E}^C(\Omega)$. The next definition shows how the space of distributions on Ω is to be embedded into $\mathcal{E}^J(\Omega)$ resp. $\mathcal{E}^C(\Omega)$.

5.5 Definition. *For $u \in \mathcal{D}'(\Omega)$, define*

$$\begin{aligned} &\iota^J : \mathcal{D}'(\Omega) \to \mathcal{E}^J(\Omega) && (\iota^J u)(\varphi, x) := \langle u, \varphi \rangle \\ &\iota^C : \mathcal{D}'(\Omega) \to \mathcal{E}^C(\Omega) && (\iota^C u)(\varphi, x) := \langle u, \varphi(. - x) \rangle \end{aligned}$$

By definition, $\iota^C = T^* \circ \iota^J$.

It remains to introduce the respective extensions of partial differentiation from $\mathcal{D}'(\Omega)$ to $\mathcal{E}^C(\Omega)$ resp. $\mathcal{E}^J(\Omega)$ and the respective actions of a diffeomorphism.

5.6 Definition. *For $i = 1, \ldots, s$, define*

$$\begin{array}{ll} D_i^C : \mathcal{E}^C(\Omega) \to \mathcal{E}^C(\Omega) & \qquad D_i^C := \partial_i, \\ D_i^J : \mathcal{E}^J(\Omega) \to \mathcal{E}^J(\Omega) & \qquad D_i^J := (T^*)^{-1} \circ \partial_i \circ T^*, \end{array}$$

i.e., for $R \in \mathcal{E}^J(\Omega)$, $(\varphi, x) \in \mathcal{A}_0(\Omega) \times \Omega$ we set

$$(D_i^J R)(\varphi, x) := -((\mathrm{d}_1 R)(\varphi, x))(\partial_i \varphi) + (\partial_i R)(\varphi, x).$$

Of course we have to demonstrate that for given $R \in \mathcal{E}^C(\Omega)$ and $(\varphi, x) \in U(\Omega)$, $(D_i^C R)(\varphi, x)$ in fact exists and that $(\varphi, x) \mapsto (D_i^C R)(\varphi, x)$ is smooth on $U(\Omega)$ with respect to τ_2 (and similar for $R \in \mathcal{E}^J(\Omega)$ and D_i^J). This being a non-trivial task—in particular for the case of the innocent-looking map $D_i^C = \partial_i$ [*sic*!]—requiring some technical prerequisites, we have to defer it to the following section.

Commutativity of the following diagram is immediate:

$$\begin{array}{ccc} \mathcal{D}'(\Omega) & \xrightarrow{\partial_i} & \mathcal{D}'(\Omega) \\ \Big\downarrow \iota^C & & \Big\downarrow \iota^C \\ \mathcal{E}^C(\Omega) & \xrightarrow{D_i^C} & \mathcal{E}^C(\Omega) \\ \Big\uparrow T^* & & \Big\uparrow T^* \\ \mathcal{E}^J(\Omega) & \xrightarrow{D_i^J} & \mathcal{E}^J(\Omega) \end{array}$$

5.7 Definition. *Let $\mu : \tilde{\Omega} \to \Omega$ be a diffeomorphism. Define*

$$\begin{array}{rccc} \bar{\mu}^J : & \mathcal{A}_0(\tilde{\Omega}) \times \tilde{\Omega} & \to & \mathcal{A}_0(\Omega) \times \Omega \\ \bar{\mu}^C : & U(\tilde{\Omega}) & \to & U(\Omega) \\ \bar{\mu}_\varepsilon : & U_\varepsilon(\tilde{\Omega}) & \to & U_\varepsilon(\Omega) \end{array}$$

by

$$\begin{array}{rcl} \bar{\mu}^J(\tilde{\varphi}, \tilde{x}) & := & \big((\tilde{\varphi} \circ \mu^{-1}) \cdot |\det D\mu^{-1}|\,,\, \mu\tilde{x}\big)\,, \\ \bar{\mu}^C(\tilde{\varphi}, \tilde{x}) & := & \big(T^{-1} \circ \bar{\mu}^J \circ T\big)\,(\tilde{\varphi}, \tilde{x}) \\ & = & \big(\tilde{\varphi}(\mu^{-1}(. + \mu\tilde{x}) - \tilde{x}) \cdot |\det D\mu^{-1}(. + \mu\tilde{x})|\,,\, \mu\tilde{x}\big)\,. \\ \bar{\mu}_\varepsilon(\tilde{\varphi}, \tilde{x}) & := & \big((S^{(\varepsilon)})^{-1} \circ T^{-1} \circ \bar{\mu}^J \circ T \circ S^{(\varepsilon)}\big)\,(\tilde{\varphi}, \tilde{x}) \\ & = & \left(\tilde{\varphi}\left(\dfrac{\mu^{-1}(\varepsilon. + \mu\tilde{x}) - \tilde{x}}{\varepsilon}\right) \cdot |\det D\mu^{-1}(\varepsilon. + \mu\tilde{x})|\,,\, \mu\tilde{x}\right). \end{array}$$

5.8 Definition. *Let $\mu : \tilde\Omega \to \Omega$ be a diffeomorphism and $\varepsilon \in I$. Define*

$$\begin{array}{rccc} \hat\mu^J : & \mathcal{E}^J(\Omega) & \to & \mathcal{E}^J(\tilde\Omega) \\ \hat\mu^C : & \mathcal{E}^C(\Omega) & \to & \mathcal{E}^C(\tilde\Omega) \\ \hat\mu_\varepsilon : & \mathcal{C}^\infty(U_\varepsilon(\Omega)) & \to & \mathcal{C}^\infty(U_\varepsilon(\tilde\Omega)) \end{array}$$

by $\hat\mu^J := (\bar\mu^J)^*$, $\hat\mu^C := (\bar\mu^C)^*$, $\hat\mu_\varepsilon := (\bar\mu_\varepsilon)^*$, i.e.,

$$\begin{array}{rcll} (\hat\mu^J R)(\tilde\varphi,\tilde x) & := & R(\bar\mu^J(\tilde\varphi,\tilde x)) & (R \in \mathcal{E}^J(\Omega), \quad (\tilde\varphi,\tilde x) \in \mathcal{A}_0(\tilde\Omega)\times\tilde\Omega), \\ (\hat\mu^C R)(\tilde\varphi,\tilde x) & := & R(\bar\mu^C(\tilde\varphi,\tilde x)) & (R \in \mathcal{E}^C(\Omega), \quad (\tilde\varphi,\tilde x) \in U(\tilde\Omega)), \\ (\hat\mu_\varepsilon R)(\tilde\varphi,\tilde x) & := & R(\bar\mu_\varepsilon(\tilde\varphi,\tilde x)) & (R \in \mathcal{C}^\infty(U_\varepsilon(\Omega)), \ (\tilde\varphi,\tilde x) \in U_\varepsilon(\tilde\Omega)). \end{array}$$

For $X \in \{C, J\}$ we obtain

$$\begin{tikzcd} \mathcal{D}'(\Omega) \arrow[r, "\mu^*"] \arrow[d, "\iota^X"] & \mathcal{D}'(\tilde\Omega) \arrow[d, "\iota^X"] \\ \mathcal{E}^X(\Omega) \arrow[r, "\hat\mu^X"] & \mathcal{E}^X(\tilde\Omega) \end{tikzcd}$$

where for $u \in \mathcal{D}'(\Omega)$, $\mu^* u$ is defined by $\langle \mu^* u, \varphi\rangle := \langle u, (\varphi\circ\mu^{-1})\cdot|\det D\mu^{-1}|\rangle$ ($\varphi \in \mathcal{D}(\Omega)$) which extends $f \mapsto \mu^* f = f\circ\mu$ for $f \in \mathcal{C}^\infty(\Omega)$.

$$\begin{tikzcd} U_\varepsilon(\tilde\Omega) \arrow[r, "\bar\mu_\varepsilon"] \arrow[d, "S^{(\varepsilon)}"] & U_\varepsilon(\Omega) \arrow[d, "S^{(\varepsilon)}"] \\ U(\tilde\Omega) \arrow[r, "\bar\mu^C"] \arrow[d, "T"] & U(\Omega) \arrow[d, "T"] \\ \mathcal{A}_0(\tilde\Omega)\times\tilde\Omega \arrow[r, "\bar\mu^J"] & \mathcal{A}_0(\Omega)\times\Omega \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}^\infty(U_\varepsilon(\tilde\Omega)) & \mathcal{C}^\infty(U_\varepsilon(\Omega)) \arrow[l, "\hat\mu_\varepsilon"'] \\ \mathcal{E}^C(\tilde\Omega) \arrow[u, "(S^{(\varepsilon)})^*"] & \mathcal{E}^C(\Omega) \arrow[l, "\hat\mu^C"'] \arrow[u, "(S^{(\varepsilon)})^*"] \\ \mathcal{E}^J(\tilde\Omega) \arrow[u, "T^*"] & \mathcal{E}^J(\Omega) \arrow[l, "\hat\mu^J"'] \arrow[u, "T^*"] \end{tikzcd}$$

Definitions 5.7 and 5.8 reflect the fact that in Definition **(D5)** of section 3, we chose to regard distributions (and, in the sequel, non-linear generalized functions) as generalizations of *functions*, acting as functionals on test densities (compare, e.g., [27]). This approach has to be distinguished from constructing distributions as distributional *densities*, acting on test functions (see, e.g., [20]).

In the following table, we compare the C-formalism and the J-formalism regarding simplicity of the respective definitions and, in the last item, the degree of familiarity.

C-formalism	Feature	J-formalism
$-$	domain of basic space $\mathcal{E}(\Omega)$	$+$
$-$	smoothness structure	$+$
$-$	embedding of $\mathcal{D}'(\Omega)$	$+$
$+$	formula for differentiation	$-$
$+$	solving differential equations	$-$
$-$	action induced by a diffeomorphism	$+$
$+$	formula for testing	$-$
$-$	generalization to manifolds	$+$
$+$	tradition	$-$

The distribution of the $+$'s and $-$'s in the table should be rather obvious by inspecting the corresponding definitions. Due to the absence of a linear structure on a general smooth manifold, it is clear that the C-formalism does not lend itself to a definition of non-linear generalized functions on manifolds based only on intrinsic terms, whereas the J-formalism in fact does permit such a construction; see [26].

We conclude this section by presenting an example that emphasizes the importance of carefully distinguishing between the topologies τ_1 and τ_2 on $U(\Omega)$.

5.9 Example. We will specify an open subset Ω of $\mathbb{R}^2$, a line segment of the form $\Phi(t) := (\varphi, z) + t(\psi, v)$ $(-1 \leq t \leq 1)$ in $U(\Omega)$ and a distribution u on Ω such that

$$\lim_{t\to 0} \frac{(\iota^C u)(\Phi(t)) - (\iota^C u)(\Phi(0))}{t} = \infty.$$

This seems to suggest that in the point (φ, z), the function $\iota^C u$ on $U(\Omega)$ (which ought to be smooth according to our definitions) has no directional derivative with respect to (ψ, v); or, to phrase it differently, that the composition of the functions $\iota^C u$ and Φ (both of which have the appearence of being smooth) is not even differentiable in (φ, z). We will leave the solution to this puzzle for the end of the example. First we give the details of the construction.

Let $\Omega := \{(x,y) \in \mathbb{R}^2 \mid x > -y^2 - 1\}$, $z := (0,0)$, $v := (0,1)$. Choose $\rho_1 \in \mathcal{D}(\mathbb{R})$ such that $\operatorname{supp}\rho_1 \subseteq [0, \frac{3}{2}]$ and $\rho_1(x) = \exp(-\frac{1}{x})$ on I. Let $c := \int \rho_1$ and choose $\rho_2 \in \mathcal{D}(\mathbb{R})$ such that $\operatorname{supp}\rho_2 \subseteq [\frac{3}{2}, 2]$ and $\int \rho_2 = 1$. Then $\rho := \rho_1 - c\rho_2$ has its support contained in $[0,2]$, coincides with $\exp(-\frac{1}{x})$ on I and satisfies $\int \rho = 0$. Pick $\omega \in \mathcal{D}(\mathbb{R})$ with the properties $\operatorname{supp}\omega = [-2, +2]$, $0 \leq \omega \leq 1$ and $\omega \equiv 1$ on $[-1, +1]$. Now define $\psi \in \mathcal{D}(\mathbb{R}^2)$ by $\psi(x,y) := \omega(y) \cdot \rho(x + 1 - y^2)$.

Finally, in order to obtain Φ as defined above, take any $\varphi \in \mathcal{A}_0(\mathbb{R}^2)$ whose support is located at the right hand side of the line given by $x = 6$. It is easy to check that

$\Phi(t) = (\varphi + t\psi, z + tv)$ belongs to $U(\Omega)$ for all $t \in [-1, +1]$.

Now there is still u to be defined. To this end, let

$$f(x) := \begin{cases} \frac{1}{x^2}\exp(\frac{1}{x} + \frac{2}{x^2}) & (0 < x < 1) \\ 0 & (x \geq 1) \end{cases} .$$

For $\sigma \in \mathcal{D}(\Omega)$ define the distribution $u \in \mathcal{D}'(\Omega)$ by

$$\langle u, \sigma \rangle := \int_{-1}^{0} f(x+1)\sigma(x,0)\,dx = \int_{0}^{1} f(x)\sigma(x-1,0)\,dx.$$

For $0 < |t| \leq 1$, it follows

$$\frac{1}{t}[(\iota^C u)(\varphi + t\psi, z + tv) - (\iota^C u)(\varphi, z)] = \frac{1}{t} t \langle u, \psi(x, y-t) \rangle = \int_{0}^{1} f(x)\rho(x - t^2)\,dx.$$

We will show that for $0 < |t| \leq \frac{1}{\sqrt{2}}$, the value of the last integral can be estimated from below by $\exp(\frac{1}{t^2} - 1) - \exp(1)$, thus tending to infinity for $t \to 0$. Substituting $x = \frac{1}{u}$, $t^2 = \frac{1}{v}$, we obtain

$$\begin{aligned} \int_{0}^{1} f(x)\rho(x-t^2)\,dx &= \int_{1}^{v} e^{u+2u^2} e^{-\frac{vu}{v-u}}\,du \geq \int_{1}^{v-1} e^{2u^2} e^{-\frac{u^2}{v-u}}\,du \geq \int_{1}^{v-1} e^{2u^2} e^{-u^2}\,du \\ &\geq \int_{1}^{v-1} e^{u}\,du = e^{v-1} - e = e^{\frac{1}{t^2}-1} - e. \end{aligned}$$

The apparent inconsistencies mentioned at the beginning of the example dissolve by taking into account that, in fact, both τ_1 and τ_2 are involved in the argument: The statement that $\Phi : [-1, +1] \to U(\Omega)$ is smooth is true only if it refers to τ_1 (the image of any neighborhood of 0 under Φ is even unbounded with respect to τ_2 since the supports of $T(\Phi(t))$ are not contained in any compact subset of Ω around $t = 0$). The statement that $\iota^C u$ is smooth is true only if $U(\Omega)$ is endowed with the topology τ_2 induced by the natural topology τ_Ω of $\mathcal{A}_0(\Omega) \times \Omega$ via T. τ_2 being strictly finer than τ_1, we cannot infer the differentiability of $(\iota^C u) \circ \Phi$ from the actual smoothness properties of $\iota^C u$ resp. Φ. Another way of capturing the problem is by pointing out that (ψ, v) is not a member of the tangent space to $U(\Omega)$ at (φ, z) (in the sense of the following section) since $\operatorname{supp}\psi$ is not contained in $\Omega - z = \Omega$.

6 Calculus on $U_\varepsilon(\Omega)$

The purpose of this section is to develop an appropriate framework for defining and handling differentials of any order of a function $f : U_\varepsilon(\Omega) \to \mathbb{C}$ which is smooth with respect to $\tau_{2,\varepsilon}$ (by definition, f is of the form $f_0 \circ T \circ S^{(\varepsilon)}$ where $f_0 \in \mathcal{C}^\infty(\mathcal{A}_0(\Omega)\times\Omega)$). By choosing $\varepsilon = 1$, this includes the case of smooth functions on $U(\Omega)$, i.e., of elements of the basic space $\mathcal{E}^C(\Omega)$. As a matter of fact, the author of [28] has payed only minor attention to these questions. However, it should be clear even from a glimpse at sections 7 and 10, in particular, that a sound definition and a proper handling of the differentials of $R_\varepsilon := R^J \circ T \circ S^{(\varepsilon)} = R^C \circ S^{(\varepsilon)}$ are crucial for the construction of a diffeomorphism invariant Colombeau algebra.

To start with, we discuss an important property of the sets $U_\varepsilon(\Omega)$ which will be fundamental in the sequel at many places. Loosely speaking, every subset of $\mathcal{A}_0(\mathbb{R}^s)\times\Omega$ which is "not too large" finally gets into $U_\varepsilon(\Omega)$ by scaling and does not feel any difference between $\tau_{1,\varepsilon}$ and $\tau_{2,\varepsilon}$. To this end, we introduce the following notation:

6.1 Definition. *For every compact subset K of Ω define*

$$
\begin{aligned}
\mathcal{A}_{0,K}(\Omega) &:= \{\varphi \in \mathcal{A}_0(\Omega) \mid \operatorname{supp}\varphi \subseteq K\},\\
\mathcal{A}_{00,K}(\Omega) &:= \{\varphi \in \mathcal{A}_{00}(\Omega) \mid \operatorname{supp}\varphi \subseteq K\},\\
U_K(\Omega) &:= T^{-1}(\mathcal{A}_{0,K}(\Omega) \times \Omega),\\
U_{\varepsilon,K}(\Omega) &:= (S^{(\varepsilon)})^{-1}(U_K(\Omega)).
\end{aligned}
$$

By definition, we have

$$
\begin{aligned}
U_K(\Omega) &= \{(\varphi,x) \in \mathcal{A}_0(\mathbb{R}^s) \times \Omega \mid \operatorname{supp}\varphi \subseteq K - x\},\\
U_{\varepsilon,K}(\Omega) &= (S^{(\varepsilon)})^{-1}T^{-1}(\mathcal{A}_{0,K}(\Omega) \times \Omega)\\
&= \{(\varphi,x) \in \mathcal{A}_0(\mathbb{R}^s) \times \Omega \mid \operatorname{supp}\varphi \subseteq \varepsilon^{-1}(K - x)\}.
\end{aligned}
$$

Then it is immediate that for $K \subset\subset \Omega$, the topologies on $\mathcal{A}_{0,K}(\Omega)$ inherited from the natural topologies of $\mathcal{A}_0(\mathbb{R}^s)$ and $\mathcal{A}_0(\Omega)$, respectively, coincide. Consequently, on $U_K(\Omega)$ the topologies τ_1 and τ_2 are equal, as are $\tau_{1,\varepsilon}$ and $\tau_{2,\varepsilon}$ on $U_{\varepsilon,K}(\Omega)$.

We now are in a position to complement Definition 5.6 by establishing that the derivation operators D_i^C and D_i^J are in fact well-defined. From the explicit formulas for D_i^C resp. D_i^J one is certainly tempted to view the former as being the simpler one of them since it does not seem to involve infinite-dimensional calculus. Yet appearances are deceiving in this case: Since we have to view $U(\Omega)$ as a manifold modelled over $\mathcal{A}_0(\Omega) \times \Omega$ the only legitimate way of interpreting $(D_i^C R^C)(\varphi,x) = (\partial_i R^C)(\varphi,x)$ is to push forward the curve $t \mapsto (\varphi, x + te_i)$ via T to $\mathcal{A}_0(\Omega) \times \Omega$

and to study the directional derivative of $R^C \circ T^{-1}$ along $c : t \mapsto T(\varphi, x + te_i) = (\varphi(\,.\, - (x + te_i)), x + te_i)$ at $t = 0$.

To this end, first note that for small absolute values of t, c actually takes values in $\mathcal{A}_0(\Omega) \times \Omega$. Moreover, $t \mapsto c(t)$ is a smooth curve in $\mathcal{A}_0(\Omega) \times \Omega$ with respect to τ_0, for the time being, according to Proposition 4.8. Since c maps some interval $[-\delta, +\delta]$ into $\mathcal{A}_{0,K}(\Omega) \times \Omega$ for a suitable $K \subset\subset \Omega$, the restriction of c to $(-\delta, +\delta)$ is smooth even with respect to τ_Ω. Therefore, the directional derivative of $R^C \circ T^{-1}$ along $c : t \mapsto T(\varphi, x + te_i) = (\varphi(\,.\, - (x + te_i)), x + te_i)$ at $t = 0$ exists. Having established existence, we can calculate its value as being given by

$$\lim_{t \to 0} \frac{1}{t}[R \circ T^{-1}(c(t)) - R \circ T^{-1}(c(0))] = \lim_{t \to 0} \frac{1}{t}[R(\varphi, x + te_i) - R(\varphi, x)].$$

Thus the usual formula works for $R^C \in \mathcal{C}^\infty(U(\Omega))$ and $D_i^C = \partial_i$, although $U(\Omega)$ is not a linear space.

Finally, to see that $\partial_i R^C$ is again smooth, we have to convince ourselves that $(\partial_i R^C) \circ T^{-1} = (T^{-1})^* D_i^C R^C = D_i^J (T^{-1})^* R^C = D_i^J (R^C \circ T^{-1})$ is smooth on $\mathcal{A}_0(\Omega) \times \Omega$. Since, by definition, $R^J := R^C \circ T^{-1}$ is smooth on $\mathcal{A}_0(\Omega) \times \Omega$, so are its differential $\mathrm{d}_1(R^C \circ T^{-1})$ and its partial derivative $\partial_i(R^C \circ T^{-1})$ on their respective domains. By Definition 5.6, $D_i^J(R^C \circ T^{-1}) = (D_i^C R^C) \circ T^{-1}$ is smooth which is equivalent to the smoothness of $D_i^C R^C$ on $U(\Omega)$. The smoothness of $D_i^J R^J$ for given $R^J \in \mathcal{E}^J(\Omega)$, on the other hand, is immediate solely by the last part of the argument given above.

Let us return to studying the sets $U_\varepsilon(\Omega)$. For the purpose of reference, the following observation is formulated as a lemma.

6.2 Lemma. *Let $K \subset\subset L \subset\subset \mathbb{R}^s$ and let B be a subset of $\mathcal{D}(\mathbb{R}^s)$ such that all $\varphi \in B$ have their supports contained in some bounded set. Then there exists $\eta > 0$ such that* $\operatorname{supp} S_\varepsilon(\varphi) \subseteq L - x$ *for all $\varepsilon \leq \eta$ and $\varphi \in B$, $x \in K$.*

Proof. Set $h := \operatorname{dist}(K, \partial L)$. Then for each $x \in K$, L contains the closed ball $\overline{B}_h(x)$ of radius h around x. If, on the other hand, the compact ball $D := \overline{B}_r(0)$ contains the supports of all $\varphi \in B$ then putting $\eta := \frac{h}{r}$ will do: We have $\operatorname{supp} S_\varepsilon(\varphi) + x \subseteq \varepsilon D + x \subseteq L$ for $\varepsilon \leq \eta$, $\varphi \in B$, $x \in K$. □

6.3 Proposition. *Let $K \subset\subset L \subset\subset \Omega$ and let B be a subset of $\mathcal{A}_0(\mathbb{R}^s)$ such that all $\varphi \in B$ have their support contained in some fixed bounded subset of $\mathbb{R}^s$. Then there exists $\eta > 0$ such that $B \times K \subseteq U_{\varepsilon,L}(\Omega)$ for all $\varepsilon \leq \eta$. In particular, $B \times K \subseteq U_\varepsilon(\Omega)$ and the restrictions of $\tau_{1,\varepsilon}$ and $\tau_{2,\varepsilon}$ to $B \times K$ are equal.*

Proof. L, K and B satisfying the assumptions of Lemma 6.2, we obtain $\eta > 0$ such that $\operatorname{supp} S_\varepsilon(\varphi) \subseteq L - x$, i.e., $(\varphi, x) \in U_{\varepsilon,L}(\Omega)$ for all $\varepsilon \leq \eta$ and $\varphi \in B$, $x \in K$. □

The fact that for small ε the topologies $\tau_{1,\varepsilon}$ and $\tau_{2,\varepsilon}$ agree on sets of the form $B \times K$ as above is crucial to get the smoothness properties right when it comes to testing for moderateness resp. negligibility, as we will see.

With these prerequisites at hand, we now are ready to introduce the tangent space of $U_\varepsilon(\Omega)$ and to define differentials of all orders of a smooth function defined on $U_\varepsilon(\Omega)$. From an abstract point of view, the tangent space of $U_\varepsilon(\Omega)$ with respect to $\tau_{2,\varepsilon}$ at the point $(\varphi, x) \in U_\varepsilon(\Omega)$ is isomorphic to $\mathcal{A}_{00}(\Omega) \times \Omega$; to the tangent vector $(\sigma, v) \in \mathcal{A}_{00}(\Omega) \times \mathbb{R}^s$ at $(\rho, x) \in \mathcal{A}_0(\Omega) \times \Omega$ there corresponds the "tangent vector" $(S_{\frac{1}{\varepsilon}} T_{-x}(\sigma + \mathrm{d}\rho \cdot v), v) \in \mathcal{A}_{00}(\mathbb{R}^s) \times \mathbb{R}^s$ at $(\varphi, x) = (T \circ S^{(\varepsilon)})^{-1}(\rho, x) = (S_{\frac{1}{\varepsilon}} T_{-x}\rho, x) \in U_\varepsilon(\Omega)$ where $\mathrm{d}\rho \cdot v$ denotes the directional derivative of ρ with respect to v. The preceding formula is obtained by taking the derivative at $t = 0$ of the smooth curve $t \mapsto (T \circ S^{(\varepsilon)})^{-1}(\rho + t\sigma, x + tv)$. In this sense, the tangent space to $U_\varepsilon(\Omega)$ at $(\varphi, x) \in U_\varepsilon(\Omega)$ can be identified with the set of all $(\psi, v) \in \mathcal{A}_{00}(\mathbb{R}^s) \times \mathbb{R}^s$ satisfying $\operatorname{supp} \psi \subseteq \frac{\Omega - x}{\varepsilon}$. Note that in this case the kinematic tangent space coincides with the operational one (the space of derivations defined on the smooth functions): In fact, by [30], 28.7., and [22], this is true for the space $\mathcal{D}(\Omega)$ (more generally, for smooth sections with compact support of vector bundles over a manifold) and hence by [22] for its complemented subspace $\mathcal{A}_0(\Omega)$.

Essentially, Proposition 6.3 also applies to tangent vectors:

6.4 Proposition. *Let $K \subset\subset L \subset\subset \Omega$ and let B, C be subsets of $\mathcal{A}_0(\mathbb{R}^s)$ resp. $\mathcal{A}_{00}(\mathbb{R}^s)$ such that all $\omega \in B \cup C$ have their supports contained in a fixed bounded subset of $\mathbb{R}^s$. Then there exists $\eta > 0$ such that $B \times K \subseteq U_{\varepsilon,L}(\Omega)$ and $C \times \mathbb{R}^s$ is contained in the tangent space to $U_\varepsilon(\Omega)$ at (φ, x) for all $(\varphi, x) \in B \times K$.*

The proof is virtually the same as for Proposition 6.3, with B now replaced by $B \cup C$; it even yields $\operatorname{supp} \psi \subseteq \frac{L - x}{\varepsilon}$ for all tangent vectors (ψ, v) with $\psi \in C$.

Now let $f : U_\varepsilon(\Omega) \to \mathbb{C}$ be a function which is smooth with respect to $\tau_{2,\varepsilon}$. Basically, $\mathrm{d}^n f$ ought to be defined on the n-fold tangent space to $U_\varepsilon(\Omega)$, that is, on

$$\mathcal{T}^n U_\varepsilon(\Omega) := \bigsqcup_{(\varphi,x) \in U_\varepsilon(\Omega)} \{(\varphi, x)\} \times \{(\psi, v) \in \mathcal{A}_{00}(\mathbb{R}^s) \times \mathbb{R}^s \mid \operatorname{supp} \psi \subseteq \frac{\Omega - x}{\varepsilon}\}^n.$$

f being assumed as smooth with respect to $\tau_{2,\varepsilon}$ by definition, we cannot use *a priori* the structure of the surrounding space $\mathcal{A}_0(\mathbb{R}^s) \times \Omega$ to define $\mathrm{d}^n f$. Instead, we will decompose $U_\varepsilon(\Omega)$ (which has to be viewed as a manifold modelled over $\mathcal{A}_0(\Omega) \times \Omega$) into a family of subsets which is characteristic for smoothness of a function with respect to $\tau_{2,\varepsilon}$ in the sense that f is smooth on $U_\varepsilon(\Omega)$ if and only if the restriction of f to any of these subsets is smooth, yet this time—due to equality of $\tau_{1,\varepsilon}$ and $\tau_{2,\varepsilon}$ on each of these subsets—either with respect to $\tau_{1,\varepsilon}$ or $\tau_{2,\varepsilon}$. This allows the calculus of

$\mathcal{A}_0(\mathbb{R}^s) \times \Omega$ to be applied to f. In particular, differentials of f of any order can be computed already from the restrictions of f to these subsets; the chain rule holds.

For the following, fix $\varepsilon \in I$. We will simply write S in place of $S^{(\varepsilon)}$.

6.5 Proposition. *For given $x \in \Omega$ and $L \subset\subset \Omega$, set $K := K_{x,L} := \frac{L-x}{\varepsilon} (\subset\subset \frac{\Omega - x}{\varepsilon})$; choose a compact set $M := M_L$ and a positive number $h := h_L$ such that $L \subset\subset M \subset\subset \Omega$ and $0 < h < \operatorname{dist}(L, \partial M)$. Finally, set $U := U_{x,L} := B_h(x)$. Then*

$$\mathcal{T}^n U_\varepsilon(\Omega) = \bigcup_{\substack{x\in\Omega \\ L\subset\subset\Omega}} \mathcal{A}_{0,K}(\mathbb{R}^s) \times U \times \big(\mathcal{A}_{00,K}(\mathbb{R}^s) \times \mathbb{R}^s\big)^n = \bigcup_{\substack{x\in\Omega \\ L\subset\subset\Omega}} \mathcal{T}^n\big(\mathcal{A}_{0,K}(\mathbb{R}^s) \times U\big). \quad (10)$$

In particular, $U_\varepsilon(\Omega) = \bigcup_{\substack{x\in\Omega \\ L\subset\subset\Omega}} \mathcal{A}_{0,K}(\mathbb{R}^s) \times U$. Moreover, each set $\mathcal{A}_{0,K}(\mathbb{R}^s) \times U$ is contained in the corresponding set $U_{\varepsilon,M}(\Omega)$.

Proof. For $y \in U$, we have $K = \frac{L+(y-x)-y}{\varepsilon} \subseteq \frac{M-y}{\varepsilon}$. Consequently, $\mathcal{A}_{0,K}(\mathbb{R}^s) \times U$ is a subset of $U_{\varepsilon,M}(\Omega) (\subseteq U_\varepsilon(\Omega))$ and any $(\psi, v) \in \mathcal{A}_{00,K}(\mathbb{R}^s) \times \mathbb{R}^s$ belongs to the tangent space of $U_\varepsilon(\Omega)$ at (φ, y) for arbitrary $(\varphi, y) \in \mathcal{A}_{0,K}(\mathbb{R}^s) \times U$. Conversely, if $(\varphi, x) \in U_\varepsilon(\Omega)$ and vectors (ψ_i, v_i) $(i = 1 \dots, n)$ tangent to $U_\varepsilon(\Omega)$ at (φ, x) are given, set $L := x + \varepsilon \cdot (\operatorname{supp} \varphi \cup \bigcup_{i=0}^{n} \operatorname{supp} \psi_i)$. Noting that $L \subset\subset \Omega$, we obtain $(\varphi, x) \in \mathcal{A}_{0,K_{x,L}}(\mathbb{R}^s) \times U_{x,L}$ and $(\psi_i, v_i) \in \mathcal{A}_{00,K_{x,L}}(\mathbb{R}^s) \times \mathbb{R}^s$. This establishes (10). The last statement of the proposition has already been shown above. $\square$

6.6 Theorem. *Let $f : U_\varepsilon(\Omega) \to \mathbb{C}$. f is smooth (with respect to $\tau_{2,\varepsilon}$) if and only if the restriction of f to every set $\mathcal{A}_{0,H}(\mathbb{R}^s) \times V$ is smooth with respect to $\tau_{1,\varepsilon}$ where $H \subset\subset \mathbb{R}^s$, V is an open subset of Ω and $\mathcal{A}_{0,H}(\mathbb{R}^s) \times V$ is contained in $U_{\varepsilon,N}(\Omega)$ for some $N \subset\subset \Omega$.*

Proof. To begin with, note that it follows from $\mathcal{A}_{0,H}(\mathbb{R}^s) \times V \subseteq U_{\varepsilon,N}(\Omega)$ that $\tau_{1,\varepsilon}$ and $\tau_{2,\varepsilon}$ agree on $\mathcal{A}_{0,H}(\mathbb{R}^s) \times V$. Assuming f to be smooth with respect to $\tau_{2,\varepsilon}$ it is now clear that its restriction to any set $\mathcal{A}_{0,H}(\mathbb{R}^s) \times V$ is smooth with respect to $\tau_{1,\varepsilon}$.

Conversely, suppose that f is smooth with respect to $\tau_{1,\varepsilon}$ on any set $\mathcal{A}_{0,K}(\mathbb{R}^s) \times U$ as defined in Proposition 6.5. Let $L_1 \subset\subset \Omega$ and $x \in \Omega$. In a first step, we show that there exists an open neighborhood V_x of x such that $(T \circ S)^{-1}(\mathcal{A}_{0,L_1}(\Omega) \times V_x) \subseteq \mathcal{A}_{0,K}(\mathbb{R}^s) \times U$ for some $K = K_{x,L}$, $U = U_{x,L}$: Choose L as to satisfy $L_1 \subset\subset L \subset\subset \Omega$ and define K, M, h, U as it had been done in Proposition 6.5; finally, set $V_x := B_r(x)$ where $r := \min(h, \operatorname{dist}(L_1, \partial L))$. For $\varphi \in \mathcal{A}_{0,L_1}(\Omega)$ and $y \in V_x$ it now follows that $y \in U$ and $\operatorname{supp} S_{\frac{1}{\varepsilon}} T_{-y} \varphi = \frac{1}{\varepsilon} \cdot (\operatorname{supp} \varphi - y) \subseteq \frac{L_1 - y}{\varepsilon} = \frac{L_1 + (x-y) - x}{\varepsilon} \subseteq \frac{L-x}{\varepsilon} = K$.

Altogether, $(T \circ S)^{-1}(\varphi, y) \in \mathcal{A}_{0,K}(\mathbb{R}^s) \times U$. By assumption and due to the inclusion relation just shown, $f \circ (T \circ S)^{-1}$ is smooth on every set $\mathcal{A}_{0,L_1}(\Omega) \times V_x$. Since $L_1 \subset\subset \Omega$ and $x \in \Omega$ have been arbitrary, $f \circ (T \circ S)^{-1}$ is smooth on the whole of $\mathcal{A}_0(\Omega) \times \Omega$ according to (an obvious modification of) Theorem 4.1. By definition, f is smooth.

$\square$

An inspection of the preceding proof shows that it is even sufficient for the smoothness of f that the condition stated in the theorem is satisfied for all $H = K_{x,L}$, $V = U_{x,L}$ where $H = K_{x,L}$, $V = U_{x,L}$ are defined as in Proposition 6.5.

Theorem 6.6 allows us to define $\mathrm{d}^n f$ on every set $\mathcal{A}_{0,H}(\mathbb{R}^s) \times V$ as above which is contained in some set of the form $U_{\varepsilon,N}(\Omega)$ with $N \subset\subset \Omega$. Since the sets $\mathcal{A}_{0,H}(\mathbb{R}^s) \times V$ cover $U_\varepsilon(\Omega)$, the differentials of f are defined globally on $U_\varepsilon(\Omega)$. Note that, in fact, they are well-defined: Let $(\varphi, x); (\psi_1, v_1), \ldots, (\psi_n, v_n)$ be a set of data with

$$(\varphi, x) \in U_\varepsilon(\Omega),\ v_i \in \mathbb{R}^s,\ \psi_i \in \mathcal{A}_{00}(\mathbb{R}^s),\ \operatorname{supp} \psi_i \subseteq \frac{\Omega - x}{\varepsilon} \quad (i = 1, \ldots, n), \tag{11}$$

which is a member of $\mathcal{T}^n\big(\mathcal{A}_{0,H_j}(\mathbb{R}^s) \times V_j\big)$ both for $j = 0$ and 1; then either way of restricting f to $\mathcal{A}_{0,H_j}(\mathbb{R}^s) \times V_j$ gives the same value for $\mathrm{d}^n f$ evaluated at these data as the restriction of f to $\mathcal{A}_{0,H}(\mathbb{R}^s) \times V$ would produce, where $H := H_1 \cap H_2$ and $V := V_1 \cap V_2$. In the particular case where f is the restriction of some $\tilde{f} \in \mathcal{C}^\infty(\mathcal{A}_0(\mathbb{R}^s) \times \Omega)$ to $U_\varepsilon(\Omega)$ then, of course, the differentials of f will agree with the restriction of the differentials of $\tilde{f}$ to the corresponding tangent spaces.

Now it is clear that the chain rule holds for any composition of the form $f \circ \Phi$ where $\Phi : W \to U_\varepsilon(\Omega)$ is a smooth function (no matter if with respect to $\tau_{1,\varepsilon}$ or $\tau_{2,\varepsilon}$) such that the domain W of Φ can be covered by a family W_ι of open sets with the property that for each ι, $\Phi(W_\iota)$ is a subset of a suitable set $\mathcal{A}_{0,H}(\mathbb{R}^s) \times V$ being, in turn, contained in some $U_{\varepsilon,N}(\Omega)$. This is exactly the situation we will meet in section 7 when constructing the diffeomorphism invariant Colombeau algebra.

6.7 Remark. In this final remark we drop the assumption of $\varepsilon \in I$ being fixed: We demonstrate that for $R_\varepsilon := R \circ S^{(\varepsilon)}$ with $R \in \mathcal{E}^C(\Omega) = \mathcal{C}^\infty(U(\Omega))$ and for a given family of sets of data $(\varphi, x); (\psi_1, v_1), \ldots, (\psi_n, v_n)$ for which the supports of all $\varphi \in \mathcal{A}_0(\mathbb{R}^s)$ and $\psi_1, \ldots, \psi_n \in \mathcal{A}_{00}(\mathbb{R}^s)$ occurring in this family are contained in a fixed bounded set and x ranges over some compact subset of Ω (v_i being arbitrary from $\mathbb{R}^s$), $\mathrm{d}^n R_\varepsilon(\varphi, x)((\psi_1, v_1), \ldots (\psi_n, v_n))$ is defined for all suffciently small ε. This will be the typical situation in the applications which are to come along in the sequel. To this end, let a subset B of $\mathcal{D}(\mathbb{R}^s)$ be given such that all $\omega \in B$ have their support contained in some fixed bounded set, say, a closed ball $D := \overline{B}_r(0)$. Choose L satisfying $K \subset\subset L \subset\subset \Omega$ and let $0 < \varepsilon_0 < \frac{1}{r} \operatorname{dist}(L, \partial\Omega)$. For $\varepsilon \leq \varepsilon_0$, set $M := L + \overline{B}_{r\varepsilon}(0)$; we have $M \subset\subset \Omega$. Then $\mathcal{A}_{0,D}(\mathbb{R}^s) \times L^\circ \subseteq U_{\varepsilon,M}(\Omega)$ since

$\operatorname{supp}\varphi \subseteq D$ and $x \in L$ imply $\operatorname{supp} T_x S_\varepsilon \varphi \in \overline{B}_{r\varepsilon}(x) \subseteq M$. Consequently, R_ε is smooth with respect to $\tau_{1,\varepsilon}$ on $\mathcal{A}_{0,D}(\mathbb{R}^s) \times L^\circ$. Moreover, $\mathcal{A}_{00,D}(\mathbb{R}^s)$ is a subset of the tangent space of $U_\varepsilon(\Omega)$ at $(\varphi, x) \in \mathcal{A}_{0,D}(\mathbb{R}^s) \times L$. Hence we conclude that for all $\varepsilon \le \varepsilon_0$, $\mathrm{d}^n R_\varepsilon(\varphi, x)((\psi_1, v_1), \ldots (\psi_n, v_n))$ is defined for all $x \in K$ (or even $x \in L^\circ$), $\varphi \in B \cap \mathcal{A}_0(\mathbb{R}^s)$, $\psi_1, \ldots, \psi_n \in B \cap \mathcal{A}_{00}(\mathbb{R}^s)$ and $v_1, \ldots, v_n \in \mathbb{R}^s$.

7 Construction of a diffeomorphism invariant Colombeau algebra

The aim of this section is to complete Jelínek's approach to constructing a diffeomorphism invariant Colombeau algebra, guided by the blueprint sketched in section 3. Contrary to [28], we base our presentation on the C-formalism—for the convenience of those readers who are acquainted best with the notation used in [10] and [13]. Nevertheless, at each stage it should be possible without difficulty to switch to the J-formalism of [28], due to the translation apparatus described in section 5.

Apart from closing a gap in Jelínek's construction we supply those parts of the respective arguments which have been included neither in [13] nor in [28], yet which—according to our view—deserve due attention to be given. This applies, in particular, to **(T4)**, **(T5)** and **(T6)**. We also include a counterexample showing that, in particular, the question of smoothness of the objects involved in the construction in fact requires a careful treatment.

7.1 The basis for the definition of the algebra

The "basic space" $\mathcal{E}^C(\Omega) = \mathcal{C}^\infty(U(\Omega))$ and the embedding $\iota^C : \mathcal{D}'(\Omega) \to \mathcal{E}^C(\Omega)$ have already been introduced in Definition 5.5. To complete **(D1)**, it remains to introduce σ:

7.1 Definition. *Let $\sigma : \mathcal{C}^\infty(\Omega) \to \mathcal{E}^C(\Omega)$ be the map defined by*

$$(\sigma(f))(\varphi, x) := f(x) \qquad\qquad (f \in \mathcal{C}^\infty(\Omega),\ (\varphi, x) \in U(\Omega)).$$

Also, (**(D2)**) has already been taken care of in Definition 5.6. In the next step, we define the subspaces of moderate and negligible members of $\mathcal{E}^C(\Omega)$, respectively. From now on we will make free use of the convention that each (in)equality (E) involving $R(S_\varepsilon\varphi, x)$ with any arguments in place of φ and x is to be understood as "$R(S_\varepsilon\varphi, x)$ is defined [i.e., $(\varphi, x) \in U_\varepsilon(\Omega)$, i.e., $\operatorname{supp} \frac{1}{\varepsilon^s}\varphi\left(\frac{\cdot}{\varepsilon}\right) \subseteq \Omega - x$] and (E) holds".

7.2 Definition. **(D3)** ([28], 8.) *Let* $R \in \mathcal{E}^C(\Omega)$. *R is called* moderate *if the following condition is satisfied:*

$$\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^s\ \exists N \in \mathbb{N}\ \forall \phi \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s)):$$

$$\partial^\alpha(R(S_\varepsilon \phi(\varepsilon, x), x)) = O(\varepsilon^{-N}) \qquad (\varepsilon \to 0)$$

uniformly for $x \in K$. *The set of all moderate elements of* $R \in \mathcal{E}^C(\Omega)$ *will be denoted by* $\mathcal{E}_M^C(\Omega)$.

There are several (mutually equivalent) variants of the above condition defining moderateness; six of them are listed below in Theorem 10.5. The formulation in Definition 7.2 is condition (C) in that theorem. Actually, Jelínek has chosen condition (A) of Theorem 10.5 for defining moderateness in [28], 8.

7.3 Definition. **(D4)** ([28], 18.)
Let $R \in \mathcal{E}_M^C(\Omega)$. *R is called* negligible *if the following condition (which, following [28], will be denoted by* (3°)*) is satisfied:*

$$\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^s\ \forall n \in \mathbb{N}\ \exists q \in \mathbb{N}\ \forall \phi \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_q(\mathbb{R}^s)):$$

$$\partial^\alpha(R(S_\varepsilon \phi(\varepsilon, x), x)) = O(\varepsilon^{n}) \qquad (\varepsilon \to 0)$$

uniformly for $x \in K$. *The set of all negligible elements of* $R \in \mathcal{E}^C(\Omega)$ *will be denoted by* $\mathcal{N}^C(\Omega)$.

Observe that in the preceding definition R is presupposed to be moderate in the sense of Definition 7.2.

Also for the condition given in Definition 7.3 (without assuming R to be moderate) there are several equivalent reformulations (see Theorem 10.6).

At this point it might be useful to observe that in the framework of the J-formalism, the tests of **(D3)** and **(D4)** have to be performed with $R(T_x S_\varepsilon \phi(\varepsilon, x), x)$ in place of $R(S_\varepsilon \phi(\varepsilon, x), x)$, due to T^* being the appropriate bijection between $\mathcal{E}^J(\Omega)$ and $\mathcal{E}^C(\Omega)$: If $R^C = T^* R^J$ then $R^C(S_\varepsilon \phi(\varepsilon, x), x) = R^J(T_x S_\varepsilon \phi(\varepsilon, x), x)$; thus R^C is moderate resp. negligible in the C-frame if and only if R^J is moderate resp. negligible in the J-frame.

In order to make **(D3)** and **(D4)** meaningful, $R(S_\varepsilon \phi(\varepsilon, x), x)$ has to depend in a smooth way on x. This statement, though looking innocent at first glance, hides a rather delicate question: The assumptions on R resp. ϕ to be smooth refer to different topologies resp. bornologies. In fact, as Example 7.18 shows, it can happen that $R(S_\varepsilon \phi(\varepsilon, x), x)$ is not even locally bounded as a function of x for fixed, yet

arbitrarily small ε. This aspect has been treated neither in [13] nor in [28]. To put things right, the following argument is needed:

Given $K \subset\subset \Omega$, choose L such that $K \subset\subset L \subset\subset \Omega$. From the boundedness of ϕ we conclude that all $\phi(\delta, x)$ with $\delta \in I$, $x \in L$ have their support contained in a suitable fixed bounded subset of $\mathbb{R}^s$. According to Proposition 6.3 there exists $\eta > 0$ such that for all $\varepsilon \leq \eta$, $W := \{\phi(\delta, x) \mid \delta \in I, x \in L\} \times L$ is a subset of $U_\varepsilon(\Omega)$ and the respective restrictions of $\tau_{1,\varepsilon}$ and $\tau_{2,\varepsilon}$ to W are equal. Consequently, also the restrictions of τ_1 and τ_2 to $S^{(\varepsilon)}(W) = \{S_\varepsilon\phi(\delta, x) \mid \delta \in I, x \in L\} \times L$ agree. Let L° denote the interior of L. On the set $(0, \eta) \times L^\circ$ (which is open in $I \times \Omega$ and contains $(0, \frac{\eta}{2}] \times K$), the map $(\varepsilon, x) \mapsto (S_\varepsilon\phi(\varepsilon, x), x)$ is smooth with respect to τ_1 by assumption, hence also with respect to τ_2. By definition, $R \in \mathcal{C}^\infty(U(\Omega))$ amounts to R being smooth with respect to τ_2. Setting $\varepsilon_0 := \frac{\eta}{2}$, we obtain that $R(S_\varepsilon\phi(\varepsilon, x), x)$ is a smooth function of (ε, x) on the open neighborhood $(0, \eta) \times L^\circ$ of $(0, \varepsilon_0] \times K$ which, finally, makes the test conditions in **(D3)** and **(D4)** actually meaningful.

In this sense, we have to extend the convention we made immediately preceding **(D3)** by requiring that whenever derivatives of a term like $R(S_\varepsilon\phi(\varepsilon, x), x)$ on a set $(0, \varepsilon_0] \times K$ are under consideration, it is to be understood that ε_0 is sufficiently small as to make sure that $(\varepsilon, x) \mapsto R(S_\varepsilon\phi(\varepsilon, x), x)$ is smooth on an open neighborhood of $(0, \varepsilon_0] \times K$.

In section 3, definitions **(D3)** and **(D4)** have been viewed as "tests" to be performed on elements R of $\mathcal{E}^C(\Omega)$, investigating their behaviour on so-called "test objects" $\phi(\varepsilon, x)$. In the case at hand, the latter take the form of smooth bounded (in the sense of section 2) maps from $I \times \Omega$ into $\mathcal{A}_0(\mathbb{R}^s)$ for testing moderateness of R resp. into $\mathcal{A}_q(\mathbb{R}^s)$ for testing negligibility of R.

Returning to the exclusive use of the C-formalism, we will drop the superscript C in ι^C, D_i^C, $\mathcal{E}^C(\Omega)$, $\mathcal{N}^C(\Omega)$ and $\mathcal{E}_M^C(\Omega)$ from now on. Moreover, note that in the sequel, by $\partial_i = \frac{\partial}{\partial x_i}$ we will always denote the corresponding derivative with respect to x, i.e., for example, $\partial_i\phi(\varepsilon, x) = \frac{\partial}{\partial x_i}\phi(\varepsilon, x)$ which must not be confused with $\frac{\partial}{\partial \xi_i}\phi(\varepsilon, x)(\xi)$.

7.4 Theorem. (T1)

$$
\begin{array}{rlrl}
(i) & \iota(\mathcal{D}'(\Omega)) \subseteq \mathcal{E}_M(\Omega) & (ii) & \sigma(\mathcal{C}^\infty(\Omega)) \subseteq \mathcal{E}_M(\Omega) \\
(iii) & (\iota - \sigma)(\mathcal{C}^\infty(\Omega)) \subseteq \mathcal{N}(\Omega) & (iv) & \iota(\mathcal{D}'(\Omega)) \cap \mathcal{N}(\Omega) = \{0\}.
\end{array}
$$

Proof. Since this theorem does not occur explicitly in [28], we include a proof; we will be more explicit on those aspects which are new, compared to Colombeau algebras already known, and more concise concerning the rest. To start with, let $u \in \mathcal{D}'(\Omega)$, $\phi \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$ and let $K \subset\subset L \subset\subset \Omega$. By the boundedness of ϕ, there exists a bounded subset C of $\mathbb{R}^s$ such that $\operatorname{supp}\phi(\varepsilon, x) \subseteq C$ for all $\varepsilon \in I$,

$x \in L$. Consequently, for $x \in K$, $\operatorname{supp} \partial^\alpha \phi(\varepsilon, x) \subseteq C$ for all $\alpha \in \mathbb{N}_0^s$, $\varepsilon \in I$. Since for ε sufficiently small (say, for $\varepsilon \leq \varepsilon_0$), even $K + \varepsilon C$ is contained in L°, we obtain that $\operatorname{supp} \phi(\varepsilon, x)\left(\frac{.-x}{\varepsilon}\right) \subset\subset L^\circ$ for $\varepsilon \leq \varepsilon_0$, $x \in K$. Thus for the values taken by

$$\partial^\alpha((\iota u)(S_\varepsilon \phi(\varepsilon, x), x)) = \left\langle u, \partial^\alpha \frac{1}{\varepsilon^s} \phi(\varepsilon, x)\left(\frac{.-x}{\varepsilon}\right)\right\rangle$$

on $(0, \varepsilon_0] \times K$, only the restriction of u to L° is relevant. Moreover, again by the boundedness of ϕ, each $\partial^\alpha \frac{1}{\varepsilon^s} \phi(\varepsilon, x)\left(\frac{y-x}{\varepsilon}\right)$ is of order at most $\varepsilon^{-|\alpha|-s}$ as $\varepsilon \to 0$, uniformly for $x \in K$, $y \in \mathbb{R}^s$. Finally, integrating the modulus of the latter function over $\mathbb{R}^s$ with respect to y gives values of order at most $\varepsilon^{-|\alpha|}$, uniformly for $x \in K$.

(i) Consider first the case $f \in \mathcal{C}(\Omega)$. Then for $\varepsilon \leq \varepsilon_0$, $|\partial^\alpha((\iota f)(S_\varepsilon \phi(\varepsilon, x), x))|$ is majorized by

$$\sup_L |f| \cdot \int_\Omega \left|\partial^\alpha \frac{1}{\varepsilon^s} \phi(\varepsilon, x)\left(\frac{y-x}{\varepsilon}\right)\right| dy = O(\varepsilon^{-|\alpha|})$$

uniformly for $x \in K$. Since locally every distribution is a derivative of a suitable continuous function, a similar estimate establishes the first inclusion.

(ii) Let $f \in \mathcal{C}^\infty(\Omega)$. Then $\partial^\alpha((\sigma f)(S_\varepsilon \phi(\varepsilon, x), x)) = \partial^\alpha f(x)$ clearly is bounded on any compact set K.

(iii) Consider $(\iota - \sigma) f$ for a given $f \in \mathcal{C}^\infty(\Omega)$. Assume, in addition to the above, that ϕ takes its values in $\mathcal{A}_q(\mathbb{R}^s)$. Then for $\varepsilon \leq \varepsilon_0$ and $x \in K$,

$$\partial^\alpha(\iota f - \sigma f)(S_\varepsilon \phi(\varepsilon, x), x)) = \sum_\beta \binom{\alpha}{\beta} \int_{\frac{\Omega - x}{\varepsilon}} \left[(\partial^\beta f)(z\varepsilon + x) - (\partial^\beta f)(x)\right] \partial^{\alpha-\beta} \phi(\varepsilon, x)(z)\, dz.$$

Taylor expansion of each $\partial^\beta f$ up to order q yields that all terms containing a power of ε less or equal to q vanish due to $\partial^{\alpha-\beta}\phi(\varepsilon, x) \in \mathcal{A}_q(\mathbb{R}^s) \cup \mathcal{A}_{q0}(\mathbb{R}^s)$. All the remainder terms are (smooth functions) of order at most ε^{q+1}, uniformly for $x \in K$, $z \in C$. Therefore, $\partial^\alpha(\iota f - \sigma f)(S_\varepsilon \phi(\varepsilon, x), x)) = O(\varepsilon^{q+1})$.

(iv) Suppose $\iota u \in \mathcal{N}(\Omega)$ for some $u \in \mathcal{D}'(\Omega)$. For $K \subset\subset \Omega$ choose $q \in \mathbb{N}$ such that the condition in **(D4)** is satisfied for $\alpha = 0$, $n = 1$. Pick any $\varphi \in \mathcal{A}_q(\mathbb{R}^s)$ and set $\phi(\varepsilon, x) := \varphi$ for all $\varepsilon \in I$, $x \in \Omega$. Then by the negligibility of ιu, $(u * S_\varepsilon \check{\varphi})(x) = \langle u, \varepsilon^{-s} \varphi(\varepsilon^{-1}(. - x))\rangle \to 0$ as $\varepsilon \to 0$, uniformly on K. This shows that u, being the weak limit of the smooth regular distributions $(u * S_\varepsilon \check{\varphi})$, is equal to 0. □

Also, we immediately get

7.5 Theorem. **(T2)** ([28], 19.) *$\mathcal{E}_M(\Omega)$ is a subalgebra of $\mathcal{E}(\Omega)$.*

7.6 Theorem. **(T3)** ([28], 19.) *$\mathcal{N}(\Omega)$ is an ideal in $\mathcal{E}_M(\Omega)$.*

7.2 The approach taken by J. Jelínek

While the conditions given in Definitions 7.2 and 7.3 are adequate for proving **(T1)**–**(T3)**, we do need appropriate reformulations for establishing the invariance of $\mathcal{E}_M(\Omega)$ and $\mathcal{N}(\Omega)$ under differentiation (**(T4)**, **(T5)**) as well as under the action induced by a diffeomorphism (**(T6)**–**(T8)**). Suitable equivalent conditions allowing to prove **(T4)** and **(T5)**, on the one hand, are given in Theorem 17 and in part $(3°) \Leftrightarrow (2°)$ of Theorem 18 in [28], respectively. We will quote these theorems below as 7.12 and 7.13.

To establish diffeomorphism invariance, on the other hand, two problems have to be coped with: First, transformed test objects in general are not defined on the whole of $I \times \Omega$; secondly, the property $\phi(\varepsilon, x) \in \mathcal{A}_q(\mathbb{R}^s)$ (as occurring in Definition 7.3) is not preserved under the action of a diffeomorphism. The first of these aspects, though presenting considerable intricacies, is covered only by a few remarks in [28] which, in our view, do not provide a treatment as rigorous as these questions require. The appropriate reformulations of Definitions 7.2 and 7.3 dealing with the poor domains of transformed test objects are provided by (C) $\Leftrightarrow$ (Z) of Theorem 10.5 and (C$''$) $\Leftrightarrow$ (Z$''$) of Corollary 10.7, respectively. In order to cope with the problem of $\phi(\varepsilon, x) \in \mathcal{A}_q(\mathbb{R}^s)$ not being preserved by a diffeomorphism, Jelínek claims in part $(3°) \Leftrightarrow (4°)$ of [28], Theorem 18 that $R \in \mathcal{E}_M(\Omega)$ is negligible (condition $(3°)$) if and only if it passes the test on test objects ϕ having only asymptotically vanishing moments of order q on K (condition $(4°)$), as compared to $\phi(\varepsilon, x) \in \mathcal{A}_q(\mathbb{R}^s)$ required by condition $(3°)$. While $(4°) \Rightarrow (3°)$ is obvious, the converse statement is not true (see Example 7.7 below). The error in the proof of $(3°) \Rightarrow (4°)$ consists in passing from terms of the form $\mathrm{d}_1 \partial^\alpha [R_\varepsilon(\dots)]$ to $[\mathrm{d}_1 \partial^\alpha R_\varepsilon](\dots)$ without applying the chain rule with respect to the composition of R_ε with some "inner" function represented by the dots (compare the proof of Theorem 7.9 given in section 10).

As a consequence, the construction of a diffeomorphism invariant Colombeau algebra aimed at in [28] is not complete in the following sense: Eliminating condition $(4°)$ from Theorem 18 deprives one of the possibility of proving diffeomorphism invariance for the algebra at hand. If, on the other hand, $(4°)$ is accepted as defining membership in $\mathcal{N}(\Omega)$ (provided $R \in \mathcal{E}_M(\Omega)$) then the embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$ does not preserve the product of smooth functions (being considered as regular distributions) even in the most simple cases, as can be seen from part two of Example 7.7 below. To overcome this difficulty, we will present a substitute for condition $(4°)$ (see Theorem 7.9 below) which in fact is equivalent to $(3°)$ under the assumption of moderateness and, moreover, allows to deduce diffeomorphism invariance of $\mathcal{N}(\Omega)$.

7.7 Examples. (1) Let $\Omega := \mathbb{R}$ and denote by u the regular distribution on $\mathbb{R}$ defined by $\langle u, \varphi \rangle := \int \xi \varphi(\xi)\, d\xi$ ($\varphi \in \mathcal{D}(\mathbb{R})$). According to part (iii) of Theorem 7.4,

$R := \iota u - \sigma u$ is a member of $\mathcal{N}(\mathbb{R})$, that is, R is moderate and satisfies the condition specified in Definition 7.3 which is condition (3°) of [28], Theorem 18. We are going to show that R in fact violates condition (4°), thereby disproving (3°) ⇒ (4°). It is immediate from the definitions that R is given by $R(\varphi, x) := \int \xi\varphi(\xi)\, d\xi$ ($\varphi \in \mathcal{A}_0(\mathbb{R})$, $x \in \mathbb{R}$). Set K:=$\{0\}$, $\alpha := 1$ and $n := 2$. For any given $q \in \mathbb{N}$, define a test object ϕ_q by $\phi_q(\varepsilon, x) := \varphi_q + x \cdot \psi_q$ ($0 < \varepsilon \leq 1$, $x \in \mathbb{R}$) where φ_q is an arbitrary fixed member of $\mathcal{A}_q(\mathbb{R})$ and $\psi_q \in \mathcal{A}_{00}(\mathbb{R})$ is chosen as to satisfy $\int \xi\psi_q(\xi)\, d\xi = 1$. Then ϕ_q belongs to $\mathcal{C}^\infty_b(I \times \mathbb{R}, \mathcal{A}_0(\mathbb{R}))$ and, being equal to φ_q on $K = \{0\}$, has asymptotically vanishing moments of order q on K, as required by condition (4°). Yet

$$\partial(R(S_\varepsilon\phi_q(\varepsilon, x), x)) = \partial(\varepsilon \cdot x) = \varepsilon \neq O(\varepsilon^2),$$

no matter how large q is chosen. This manifestly contradicts condition (4°) for the choices of K, α, n made above. We also see that adopting (4°) (together with moderateness, of course) as defining property for $\mathcal{N}(\mathbb{R})$ would invalidate part (iii) of Theorem **(T1)** which is the basis for ι to preserve the product of smooth functions. This is made explicit by the following item.

(2) Define ϕ_q as in Example (1), yet this time requiring both $\int \xi\psi_q(\xi)\, d\xi = 1$ and $\int \xi^2\psi_q(\xi)\, d\xi = 0$ for ψ_q to be chosen from $\mathcal{A}_{00}(\mathbb{R})$ and, in addition, $\varphi_q \in \mathcal{A}_{\max(2,q)}(\mathbb{R})$. Denoting by f the identity function on $\mathbb{R}$, f can be identified with the distribution u introduced previously. A straightforward calculation yields

$$\big(\iota(f) \cdot \iota(f) - \iota(f \cdot f)\big)(\varphi, x) = \Big(\int \xi\varphi(\xi)\, d\xi\Big)^2 - \int \xi^2\varphi(\xi)\, d\xi$$

where $\varphi \in \mathcal{A}_0(\mathbb{R})$, $x \in \mathbb{R}$. Substituting $S_\varepsilon\phi_q(\varepsilon, x)$ for φ and taking second derivatives at $x = 0$, we obtain

$$\partial^2\big((\iota(f) \cdot \iota(f) - \iota(f \cdot f))(S_\varepsilon\phi_q(\varepsilon, x), x)\big)\big|_{x=0} = \partial^2\big(\varepsilon^2 \cdot x^2\big)\big|_{x=0} = 2\varepsilon^2 \neq O(\varepsilon^3)$$

for any $q \in \mathbb{N}$. Since again ϕ_q is a test object having asymptotically vanishing moments of order q on $K = \{0\}$, $\big(\iota(f) \cdot \iota(f) - \iota(f \cdot f)\big)$ does not satisfy (4°), this time with respect to $K = \{0\}$, $\alpha := 2$, $n := 3$. Consequently, adopting (4°) in place of (3°) as the defining property for negligibility would prevent the restriction of ι to $\mathcal{C}^\infty(\mathbb{R})$ from being an algebra homomorphism.

To complete the prerequisites for establishing **(T4)**–**(T8)** it remains to state the theorem replacing part (3°) ⇔ (4°) of Theorem 18 of [28]. To this end, we introduce the following terminology (which, actually, is taken from the second paper of this series):

7.8 Definition. *Let $\phi \in \mathcal{C}^\infty_b(I \times \mathbb{R}, \mathcal{A}_0(\mathbb{R}))$, $K \subset\subset \Omega$, $q \in \mathbb{N}$. ϕ is said to be of type $[\mathrm{A}^\infty_1]_{K,q}$ if all derivatives $\partial^\beta_x\phi(\varepsilon, x)$ ($\beta \in \mathbb{N}^s_0$) have asymptotically vanishing moments of order q on K.*

In the preceding definition, “A”, “l” and “∞” stand for “asymptotically vanishing moments”, “locally” (i.e., only on the particular compact set K under consideration) and “derivatives of all orders”.

7.9 Theorem. *Let $R \in \mathcal{E}_M(\Omega)$. R is negligible, i.e., R satisfies the condition specified in Definition 7.3 if and only if it satisfies the following property (which will be referred to as (4^∞)):*

$\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^s\ \forall n \in \mathbb{N}\ \exists q \in \mathbb{N}\ \forall \phi \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s)),\ \phi$ of type $[\mathrm{A}_\mathrm{l}^\infty]_{K,q}$:

$$\partial^\alpha(R(S_\varepsilon\phi(\varepsilon, x), x)) = O(\varepsilon^n) \qquad (\varepsilon \to 0)$$

uniformly for $x \in K$.

The proof of the preceding theorem is deferred to section 10. Restricting β in the definition of type $[\mathrm{A}_\mathrm{l}^\infty]_{K,q}$ to the value $0 \in \mathbb{N}_0^s$ turns condition (4^∞) into condition (4°) of Theorem 18 of [28].

7.3 Stability under differentiation

Having set up the prerequisites for the remaining part of the construction in the previous subsection we proceed to establish Theorems **(T4)** and **(T5)**.

7.10 Theorem. (T4) *For $R \in \mathcal{E}_M(\Omega)$, $\partial_i R \in \mathcal{E}_M(\Omega)$ $(i = 1, \dots, s)$.*

7.11 Theorem. (T5) *For $R \in \mathcal{N}(\Omega)$, $\partial_i R \in \mathcal{N}(\Omega)$ $(i = 1, \dots, s)$.*

Curiously enough, the preceding Theorems are not even mentioned in Jelínek's paper [28]. We regard them as highly non-trivial, however: At first glance they might seem obvious since the respective tests ask for a certain behaviour of *all* derivatives $\partial^\alpha(R(\phi(\varepsilon, x), x))$; thus, as one might be tempted to argue, the moderateness or negligibility of R implies the respective property also for each $D_i^C R = \partial_i R$. This reasoning, however, does not take into account the fact that the expression $\partial^\alpha(R(\phi(\varepsilon, x), x))$ used for testing R involves a certain sum of partial derivatives of R multiplied by partial derivatives of $\phi(\varepsilon, x)$, according to the chain rule. There is no easy relation between the respective expressions for ∂^α and $\partial_i\partial^\alpha$ which could be used to draw from the asymptoptic behaviour of the former to infer the corresponding property for the latter.

The key tools for establishing Theorems **(T4)** and **(T5)** are Jelínek's Theorem 17 and part $(2^\circ) \Leftrightarrow (3^\circ)$ of Theorem 18 in [28]. For their ingenious proofs we refer to

the original [28]. We presume that the author was completely aware of the rôle Theorems 17 and 18 had to play in that respect, yet for some reasons he decided not to address this issue.

7.12 Theorem. ([28], Theorem 17) *Let $R \in \mathcal{E}(\Omega)$. R is moderate (i.e., a member of $\mathcal{E}_M(\Omega)$) if and only if the following condition is satisfied:*

$$\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^s\ \forall k \in \mathbb{N}_0\ \exists N \in \mathbb{N}\ \forall B\,(\textit{bounded}) \subseteq \mathcal{D}(\mathbb{R}^s):$$

$$\partial^\alpha \mathrm{d}_1^k (R \circ S^{(\varepsilon)})(\varphi, x)(\psi_1, \dots, \psi_k) = O(\varepsilon^{-N}) \qquad (\varepsilon \to 0)$$

uniformly for $x \in K$, $\varphi \in B \cap \mathcal{A}_0(\mathbb{R}^s)$, $\psi_1, \dots, \psi_k \in B \cap \mathcal{A}_{00}(\mathbb{R}^s)$.

7.13 Theorem. ([28], Theorem 18, $(3^\circ) \Leftrightarrow (2^\circ)$)
Let $R \in \mathcal{E}_M(\Omega)$. R is negligible (i.e., a member of $\mathcal{N}(\Omega)$) if and only if the following condition is satisfied:

$$\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^s\ \forall k \in \mathbb{N}_0\ \forall n \in \mathbb{N}\ \exists q \in \mathbb{N}\ \forall B\,(\textit{bounded}) \subseteq \mathcal{D}(\mathbb{R}^s):$$

$$\partial^\alpha \mathrm{d}_1^k (R \circ S^{(\varepsilon)})(\varphi, x)(\psi_1, \dots, \psi_k) = O(\varepsilon^{n}) \qquad (\varepsilon \to 0)$$

uniformly for $x \in K$, $\varphi \in B \cap \mathcal{A}_q(\mathbb{R}^s)$, $\psi_1, \dots, \psi_k \in B \cap \mathcal{A}_{q0}(\mathbb{R}^s)$.

Actually, the respective conditions in Definition 7.3 and Theorem 7.13 (that is, conditions (3°) and (2°) of Theorem 18 of [28]) are equivalent even without assuming R to be moderate: The proof is similar to the proof of Theorem 17 of [28], taking into account the equivalence of conditions (A$'$) and (C$'$) of Theorem 10.6.

It is a remarkable fact that in the condition of Theorem 7.13, k, d_1^k and $\psi_1, \dots, \psi_k$ can be omitted completely (i.e., only the case $k = 0$ has to be taken into account) without changing its content, provided R is assumed to be moderate ([28], Theorem 18, $(1^\circ) \Leftrightarrow (2^\circ)$). We can interpret this heuristically as the fact that the moderateness condition takes care of the derivatives of R to be small in the limit, provided only R itself is small in the appropriate sense. A still stronger result will be shown in the second paper of this series: Provided that R is moderate, it is even possible to omit the x-derivatives in the condition of Theorem 7.13, yielding that $R \in \mathcal{E}_M(\Omega)$ is negligible if and only if the following condition is satisfied:

$$\forall K \subset\subset \Omega\ \forall n \in \mathbb{N}\ \exists q \in \mathbb{N}\ \forall B\,(\text{bounded}) \subseteq \mathcal{D}(\mathbb{R}^s): R(S_\varepsilon \varphi, x) = O(\varepsilon^n) \quad (\varepsilon \to 0)$$

uniformly for $x \in K$, $\varphi \in B \cap \mathcal{A}_q(\mathbb{R}^s)$.

Proof of (T4) and (T5). In order to derive, for example, **(T4)** from Theorem 7.12, assume $R \in \mathcal{E}(\Omega)$ to be moderate, hence to satisfy the condition in 7.12. Let

$i \in \{1, \ldots, s\}$; due to $\partial_i(R \circ S^{(\varepsilon)}) = (\partial_i R) \circ S^{(\varepsilon)}$ we obtain $\partial^\alpha \mathrm{d}_1^k((\partial_i R) \circ S^{(\varepsilon)}) = \partial^{\alpha+e_i} \mathrm{d}_1^k (R \circ S^{(\varepsilon)})$ where e_i denotes the i-th standard unit vector in $\mathbb{R}^s$. From the preceding equation it is immediate that together with the differentials of R, also the differentials of $\partial_i R$ are of order at most ε^{-N} for $\varepsilon \to 0$ in the appropriate sense. Applying Theorem 7.12 once more, we infer the moderateness of $\partial_i R$. The proof of Theorem **(T5)** proceeds along the same lines, this time using Theorem 7.13. $\square$

7.4 Diffeomorphism invariance

For any diffeomorphism $\mu : \tilde{\Omega} \to \Omega$ the requirements of **(D5)** are satisfied for $\bar{\mu}^C : U(\tilde{\Omega}) \to U(\Omega)$ and $\hat{\mu}^C : \mathcal{E}^C(\Omega) \to \mathcal{E}^C(\tilde{\Omega})$ as in Definitions 5.7 and 5.8.

Again, we shall omit the superscript C from $\bar{\mu}^C$, $\hat{\mu}^C$.

Our next task is to establish the invariance of test objects under the appropriate action induced by μ. This, of course, is at the very heart of the diffeomorphism invariance of the Colombeau algebra to be constructed. In the end, we must be able to infer the moderateness of $\hat{\mu}R$ from the moderateness of R (and, similarly, for negligibility). Unfortunately, it need not be true in a strict sense that the class of test objects $\phi(\varepsilon, x)$ as in Definitions **(D3)** resp. **(D4)** is invariant under the action of a diffeomorphism: The transformed test objects turn out to be defined not on the whole of $I \times \Omega$ necessarily, i.e., they form a larger class than the original test objects do. Due to $\hat{\mu}R = R \circ \bar{\mu}$, we must start with the (formally stronger) assumption (let us denote it by (Z)) that R is moderate even with respect to that larger class of test objects to reach the conclusion that $\hat{\mu}R$ is moderate in the sense of **(D3)** (condition (C)). However, in Theorem 10.5 we will show that (C) and (Z) are, in fact, equivalent so that, in the end, the property of moderateness is shown to be preserved under the action of a diffeomorphism.

The following heuristic calculation clearly shows which path is to be pursued: Let $\bar{\mu}_\varepsilon$ be defined as in section 5. For $\tilde{\phi} \in \mathcal{C}_b^\infty(I \times \tilde{\Omega}, \mathcal{A}_0(\mathbb{R}^s))$ given, we have to determine a function ϕ defined on a suitable subset of $I \times \Omega$ taking values in $\mathcal{A}_0(\mathbb{R}^s)$ as to satisfy the following relation:

$$
\begin{aligned}
&(\hat{\mu}R)(S_\varepsilon \tilde{\phi}(\varepsilon, \tilde{x}), \tilde{x}) = R(\bar{\mu}(S_\varepsilon \tilde{\phi}(\varepsilon, \tilde{x}), \tilde{x})) = R(\bar{\mu} S^{(\varepsilon)}(\tilde{\phi}(\varepsilon, \tilde{x}), \tilde{x})) \\
&= R(S^{(\varepsilon)}(S^{(\varepsilon)})^{-1} \bar{\mu} S^{(\varepsilon)}(\tilde{\phi}(\varepsilon, \tilde{x}), \tilde{x})) = R(S^{(\varepsilon)} \bar{\mu}_\varepsilon(\tilde{\phi}(\varepsilon, \tilde{x}), \tilde{x})) = R(S_\varepsilon \phi(\varepsilon, \mu\tilde{x}), \mu\tilde{x})
\end{aligned}
$$

where ϕ is defined implicitly by the requirement of the last equality to hold. Obviously, this is the case if and only if $(\phi(\varepsilon, x), x) = \bar{\mu}_\varepsilon(\tilde{\phi}(\varepsilon, \mu^{-1}x), \mu^{-1}x))$ which, according to 5.7, amounts to (12) in Theorem 7.14 below. To carry out the program outlined above, three aspects of ϕ have to be handled simultaneousely: domain of definition, smoothness and boundedness.

Starting with the first of these, observe that the right hand side of (12) is only defined if ξ is an element of $\frac{\Omega - x}{\varepsilon}$ whereas we would want $\xi \mapsto \phi(\varepsilon, x)(\xi)$ to be a test function on the whole of $\mathbb{R}^s$. For the convenience of the reader, we include what essentially is the argument in [28], Remark 25: The right hand side of (12) (viewed as a smooth function on $\frac{\Omega - x}{\varepsilon}$) can be extended to a smooth function on the whole of $\mathbb{R}^s$ by setting it equal to 0 for $\xi \notin \frac{\Omega - x}{\varepsilon}$, provided its support is a compact subset of $\frac{\Omega - x}{\varepsilon}$. This, in turn, is equivalent to $\tilde{\phi}(\varepsilon, \mu^{-1}x)$ having compact support contained in $\frac{\tilde{\Omega} - \mu^{-1}x}{\varepsilon}$: Indeed, $\xi \mapsto \frac{\mu^{-1}(\varepsilon\xi + x) - \mu^{-1}x}{\varepsilon}$ maps $\frac{\Omega - x}{\varepsilon}$ diffeomorphically onto $\frac{\tilde{\Omega} - \mu^{-1}x}{\varepsilon}$. Therefore, the largest natural domain of definition for ϕ is the set D of all $(\varepsilon, x) \in I \times \Omega$ for which $\operatorname{supp} \tilde{\phi}(\varepsilon, \mu^{-1}x)$ is contained in $\frac{\tilde{\Omega} - \mu^{-1}x}{\varepsilon}$, i.e., for which $(\tilde{\phi}(\varepsilon, \mu^{-1}x), \mu^{-1}x) \in U_\varepsilon(\tilde{\Omega})$. We do not know the form of D explicitly; however, due to the boundedness of the map $\tilde{\phi}$, for each given $K \subset\subset \Omega$ there exists $\varepsilon_0 > 0$ (chosen appropriately with respect to the compact set $\mu^{-1}K$ by Proposition 6.3) such that $(\tilde{\phi}(\varepsilon, \mu^{-1}x), \mu^{-1}x) \in U_\varepsilon(\tilde{\Omega})$ for all $(\varepsilon, x) \in (0, \varepsilon_0] \times K$ which amounts to $(0, \varepsilon_0] \times K$ being a subset of D. Summarizing, ϕ is defined at least on "rectangles" of the form $(0, \varepsilon_0] \times K$ as a map taking values in $\mathcal{A}_0(\mathbb{R}^s)$. This settles the problem of the domain of ϕ in a satisfactory way, as we will see shortly.

In the light of Example 5.9 as well as of Example 7.18 at the end of this section, it seems advisable to give a careful treatment also to the question of smoothness of ϕ. We defer this to the formal proof of **(T6)**.

7.14 Theorem. (T6) ([28], 25.) *Let $\mu : \tilde{\Omega} \to \Omega$ be a diffeomorphism. Let $\tilde{\phi} \in \mathcal{C}_b^\infty(I \times \tilde{\Omega}, \mathcal{A}_0(\mathbb{R}^s))$ and define $D (\subseteq I \times \Omega)$ by*

$$D := \{(\varepsilon, x) \in I \times \Omega \mid (\tilde{\phi}(\varepsilon, \mu^{-1}x), \mu^{-1}x) \in U_\varepsilon(\tilde{\Omega})\}.$$

For $(\varepsilon, x) \in D$, set $\phi(\varepsilon, x) := \mathrm{pr}_1\, \bar{\mu}_\varepsilon(\tilde{\phi}(\varepsilon, \mu^{-1}x), \mu^{-1}x))$, i.e.,

$$\phi(\varepsilon, x)(\xi) \quad := \quad \tilde{\phi}(\varepsilon, \mu^{-1}x) \left(\frac{\mu^{-1}(\varepsilon\xi + x) - \mu^{-1}x}{\varepsilon} \right) \cdot |\det D\mu^{-1}(\varepsilon\xi + x)|. \quad (12)$$

Then ϕ satisfies the requirements 1) and 2) specified for test objects in condition (Z) of Theorem 10.5.

If, in addition, all derivatives $\partial_{\tilde{x}}^\alpha \tilde{\phi}(\varepsilon, \tilde{x})$ have asymptotically vanishing moments of order q on some compact subset $\tilde{L}$ of $\tilde{\Omega}$ ($\alpha \in \mathbb{N}_0^s$) then all derivatives $\partial_x^\alpha \phi(\varepsilon, x)$ of the the function ϕ defined by (12) have asymptotically vanishing moments of order $\left[\frac{q+1}{2}\right]$ on the (compact) set $L = \mu(\tilde{L})$.

Proof. That ϕ is well-defined on D has already been shown. To establish the smoothness of ϕ on suitable open subsets of $I \times \Omega$, expand $\bar{\mu}_\varepsilon$ to obtain

$$(\phi(\varepsilon, x), x) = (S^{(\varepsilon)})^{-1} T^{-1} \bar{\mu}^J T S^{(\varepsilon)} (\tilde{\phi}(\varepsilon, \mu^{-1}x), \mu^{-1}x)).$$

In a first step, we discuss the smoothness of $\Phi(\varepsilon, x) := TS^{(\varepsilon)}(\tilde{\phi}(\varepsilon, \mu^{-1}x), \mu^{-1}x))$. Φ involves the maps μ^{-1}, $\tilde{\phi}$, S and T, all of which are smooth by the results of Proposition 4.8, provided $\mathcal{A}_0(\mathbb{R}^s)$ is endowed with the natural locally convex topology inherited from $\mathcal{D}(\mathbb{R}^s)$. Let $K \subset\subset \Omega$; we are going to show that for suitable $\varepsilon_0 > 0$ and $\tilde{M} \subset\subset \Omega$, Φ actually maps some open neighborhood of $(0, \varepsilon_0] \times K$ into $\mathcal{A}_{0,\tilde{M}}(\tilde{\Omega}) \times \tilde{\Omega}$. To this end, choose L, M such that $K \subset\subset L \subset\subset M \subset\subset \Omega$. Set $\tilde{L} := \mu^{-1}L$, $\tilde{M} := \mu^{-1}M$ and $h := \operatorname{dist}(L, \partial M)$, $\tilde{h} := \operatorname{dist}(\tilde{L}, \partial\tilde{M})$.

Due to the boundedness of $\tilde{\phi}$, there is $r \geq \tilde{h}$ (> 0) such that the supports of all $\tilde{\phi}(\varepsilon, \tilde{x})$ with $\varepsilon \in I$, $\tilde{x} \in \tilde{L}$ are contained in the closed ball $\overline{B}_r(0)$. Setting $\eta := \frac{\tilde{h}}{r}$, Proposition 6.3 and a glance at the proof of Lemma 6.2 show that $\tilde{\phi}(I \times \tilde{L}) \times \tilde{L} \subseteq U_{\varepsilon,\tilde{M}}(\tilde{\Omega})$ for all $\varepsilon \leq \eta$. In particular, for all $x \in L$ and $\varepsilon \leq \eta$,

$$(\tilde{\phi}(\varepsilon, \mu^{-1}x), \mu^{-1}x) \in U_{\varepsilon,\tilde{M}}(\Omega). \tag{13}$$

Therefore, Φ maps the open set $U := (0, \eta) \times L^\circ$ into $\mathcal{A}_{0,\tilde{M}}(\tilde{\Omega}) \times \tilde{\Omega}$. On the latter, however, the topologies τ_0 and $\tau_{\tilde{\Omega}}$ introduced in section 5 coincide. From the smoothness of the restriction of Φ to U with respect to τ_0 (which was established above) we conclude the smoothness with respect to $\tau_{\tilde{\Omega}}$. Now we are ready to go on with the proof of the smoothness of $\phi = \operatorname{pr}_1 \circ (S^{(\varepsilon)})^{-1} \circ T^{-1} \circ \bar{\mu}^J \circ \Phi$, observing that $\bar{\mu}^J$ is smooth if the domain $\mathcal{A}_0(\tilde{\Omega}) \times \tilde{\Omega}$ and the range space $\mathcal{A}_0(\Omega) \times \Omega$ carry the topologies $\tau_{\tilde{\Omega}}$ and τ_Ω, respectively. (Note that in general, $\bar{\mu}^J$ is *not* τ_0-τ_0-smooth as can be seen from Example 7.19 below.) Weakening this conclusion by replacing τ_Ω by τ_0 on $\mathcal{A}_0(\Omega) \times \Omega$ and using once more the smoothness of T and S with respect to the usual topology of $\mathcal{A}_0(\mathbb{R}^s)$, we finally obtain that for $\varepsilon_0 := \frac{1}{2}\eta$ (if $\eta = 1$ we may choose $U := I \times L^\circ$, being open in $I \times \Omega$, and $\varepsilon_0 := 1$), ϕ is smooth on the open neighborhood U of $(0, \varepsilon_0] \times K$, as claimed by condition 1) of Theorem 10.5 (Z).

For the proof of boundedness of ϕ, we extend the argument of [28], 25., Proposition. Note that, by (13) above, ϕ is defined at least on $(0, \eta] \times L$. Let $l := \max(1, \sup\{\|D\mu_{\tilde{x}}\| \mid \tilde{x} \in \tilde{M}\})$. Then for $\tilde{x} \in \tilde{L}$, we have

$$\|\tilde{x} - \tilde{y}\| \leq \tilde{h} \Rightarrow \|\mu\tilde{x} - \mu\tilde{y}\| \leq l\|\tilde{x} - \tilde{y}\|, \tag{14}$$

due to $\overline{B}_{\tilde{h}}(\tilde{x}) \subseteq \tilde{M}$. $\overline{B}_r(0)$ containing the support of every $\tilde{\phi}(\varepsilon, \tilde{x})$ for $\varepsilon \in I$, $\tilde{x} \in \tilde{L}$, we have

$$\operatorname{supp} T_{\mu^{-1}x} S_\varepsilon \tilde{\phi}(\varepsilon, \mu^{-1}x) \subseteq \overline{B}_{r\varepsilon}(\mu^{-1}x) \subseteq \tilde{M}$$

for $\varepsilon \leq \eta$, $x \in L$. Applying $\bar{\mu}^J$ we obtain, by (14),

$$\operatorname{supp} \operatorname{pr}_1 \bar{\mu}^J TS^{(\varepsilon)}(\tilde{\phi}(\varepsilon, \mu^{-1}x), \mu^{-1}x) \subseteq \overline{B}_{lr\varepsilon}(x) \cap M$$

and, finally, $\operatorname{supp} \phi(\varepsilon, x) \subseteq \overline{B}_{lr}(0)$ for $\varepsilon \leq \eta$, $x \in L$. It follows that for each $\alpha \in \mathbb{N}_0^s$, $\operatorname{supp} \partial^\alpha \phi(\varepsilon, x) \subseteq \overline{B}_{lr}(0)$ for $\varepsilon < \eta$, $x \in L^\circ$. For the boundedness of

$\{\partial^\alpha \phi(\varepsilon, x) \mid (\varepsilon, x) \in (0, \eta_1) \times L^\circ\}$ where $\eta_1 := \min(\eta, \frac{h}{rl}) = \min(\frac{\tilde{h}}{r}, \frac{h}{rl})$ it now suffices to show that for each fixed $\beta \in \mathbb{N}_0^s$,

$$\sup\{|\partial_\xi^\beta \partial_x^\alpha \phi(\varepsilon, x)(\xi)| \,\big|\, \varepsilon < \eta_1,\ x \in L,\ \|\xi\| \leq lr\}$$

is finite. For $\varepsilon < \eta_1,\ x \in L^\circ,\ \|\xi\| \leq lr$ (observe that $\operatorname{supp} \partial_x^\alpha \phi(\varepsilon, x) \subseteq \overline{B}_{lr}(0) \subseteq \frac{\Omega - x}{\varepsilon}$) $\partial_\xi^\beta \partial_x^\alpha \phi(\varepsilon, x)(\xi)$ is a sum of terms of the form

$$\partial_{\tilde{\xi}}^{\beta_0}((\partial_{\tilde{x}}^{\alpha_0} \tilde{\phi})(\varepsilon, \mu^{-1}x)) \left(\frac{\mu^{-1}(\varepsilon\xi + x) - \mu^{-1}x}{\varepsilon} \right) \cdot g_0(x) \cdot g_1(\varepsilon, x, \xi) \cdot \ \ldots\ \cdot g_p(\varepsilon, x, \xi) \cdot$$
$$\cdot \partial_\xi^{\beta'} \partial_x^{\alpha'} |\det D\mu^{-1}(\varepsilon\xi + x)| \qquad (15)$$

where g_0 is a certain product of derivatives of components of μ^{-1} (hence bounded on L) and each g_j $(j = 1, \ldots, p)$ is some derivative of some component of $\frac{\mu^{-1}(\varepsilon\xi + x) - \mu^{-1}x}{\varepsilon}$, i.e.,

$$g_j(\varepsilon, x, \xi) = \partial_\xi^{\beta_j} \partial_x^{\alpha_j} \left(\frac{\mu_{i_j}^{-1}(\varepsilon\xi + x) - \mu_{i_j}^{-1}x}{\varepsilon} \right).$$

For $\varepsilon \leq \eta_1$, $x \in L$, $\|\xi\| \leq lr$, we have $x + \varepsilon\xi \in \overline{B}_h(x) \subseteq M$. Thus the last factor in (15) is uniformly bounded, as is $\partial_{\tilde{\xi}}^{\beta_0}((\partial_{\tilde{x}}^{\alpha_0} \tilde{\phi})(\varepsilon, \mu^{-1}x))(\tilde{\xi})$ for $\varepsilon \leq 1$, $x \in L$, $\tilde{\xi} \in \mathbb{R}^s$. It remains to discuss the boundedness of the factors g_j $(j = 1, \ldots, p)$. For the sake of simplicity, we replace α_j, β_j, i_j by α, β, i. Considering first the case $|\beta| > 0$ (say, $\beta_k \geq 1$), the uniform boundedness of

$$\partial_\xi^\beta \partial_x^\alpha \frac{\mu_i^{-1}(\varepsilon\xi + x) - \mu_i^{-1}x}{\varepsilon} = \partial_\xi^{\beta - e_k} \partial_x^\alpha \left((\partial_k \mu_i^{-1})(\varepsilon\xi + x) \right)$$

on $\varepsilon \leq \eta_1$, $x \in L$, $\|\xi\| \leq lr$ is evident. If, on the other hand, $|\beta| = 0$, choose a Lipschitz constant $l_{\alpha,i}$ for $\partial_x^\alpha \mu_i^{-1}$ with respect to L, h (in the same way as l was chosen for μ with respect to $\tilde{L}, \tilde{h}$ as to satisfy (14)). It follows that

$$\left| \frac{(\partial_x^\alpha \mu_i^{-1})(\varepsilon\xi + x) - (\partial_x^\alpha \mu_i^{-1})(x)}{\varepsilon} \right| \leq \frac{l_{\alpha,i} \|\varepsilon\xi\|}{\varepsilon} \leq l_{\alpha,i} lr$$

which establishes the uniform boundedness on $\varepsilon \leq \eta_1$, $x \in L$, $\|\xi\| \leq lr$ also of this term. Replacing $\varepsilon_0 = \frac{1}{2}\eta$ by $\varepsilon_0 := \frac{1}{2}\eta_1$, we have shown altogether that ϕ is smooth and each derivative $\partial^\alpha \phi$ is bounded on the open neighborhood $(0, \eta_1) \times L^\circ$ $(\subseteq D)$ of $(0, \varepsilon_0] \times K$, as required for satisfying conditions 1) and 2) of Theorem 10.5, (Z).

Finally, assume that all derivatives $\partial_{\tilde{x}}^\alpha \tilde{\phi}(\varepsilon, \tilde{x})$ have asymptotically vanishing moments of order q on some compact subset $\tilde{L}$ of $\tilde{\Omega}$ $(\alpha \in \mathbb{N}_0^s)$. We have to show that for all $\beta \in \mathbb{N}_0^s$ satisfying $|\beta| \leq \left[\frac{q+1}{2}\right]$ and for arbitrary $\alpha \in \mathbb{N}_0^s$,

$$\langle \xi^\beta, \partial_x^\alpha \phi(\varepsilon, x)(\xi) \rangle = \partial_x^\alpha \int \left(\frac{\mu(\varepsilon\tilde{\xi} + \tilde{x}) - \mu\tilde{x}}{\varepsilon} \right)^\beta \tilde{\phi}(\varepsilon, \tilde{x})(\tilde{\xi})\, d\tilde{\xi} = O(\varepsilon^{\left[\frac{q+1}{2}\right]}) \qquad (16)$$

uniformly for $x = \mu\tilde{x} \in L$ (or $\tilde{x} = \mu^{-1}x \in \tilde{L}$, respectively). Note that the preceding equation is meaningful since there exists $\varepsilon_0 > 0$ such that all the terms occurring therein are defined for $\varepsilon \leq \varepsilon_0$, $x \in L$, $\xi \in \mathbb{R}^s$, $\tilde{\xi} \in \mathbb{R}^s$ resp. $\tilde{\xi}$ ranging over a fixed compact set containing the supports of all $\tilde{\phi}(\varepsilon, \tilde{x})$ with $\varepsilon \leq \varepsilon_0$, $\tilde{x} \in \tilde{L}$ in its interior.

Let us consider the case $\alpha = 0$ first. Expanding μ into a Taylor series up to order n at $\tilde{x}$ we may write the first factor in the integral as a finite sum of terms of the form

$$\frac{1}{\varepsilon^{|\beta|}} \frac{\partial^{\alpha_1}\mu_{i_1}(\tilde{x} + \eta_{|\alpha_1|}\theta_{i_1}\varepsilon\tilde{\xi})}{\alpha_1!} \tilde{\xi}^{\alpha_1}\varepsilon^{|\alpha_1|} \ldots \frac{\partial^{\alpha_{|\beta|}}\mu_{i_{|\beta|}}(\tilde{x} + \eta_{|\alpha_{|\beta|}|}\theta_{i_{|\beta|}}\varepsilon\tilde{\xi})}{\alpha_{|\beta|}!} \tilde{\xi}^{\alpha_{|\beta|}}\varepsilon^{|\alpha_{|\beta|}|} \tag{17}$$

where $1 \leq |\alpha_j| \leq n+1$, $0 < \theta_i < 1$, $\eta_j = 1$ if $j = n+1$ (i.e., if the respective factor is a remainder term of the Taylor series) and $\eta_j = 0$ otherwise. Observe that in the present context, by $\alpha_1, \ldots, \alpha_{|\beta|}$ we are denoting $|\beta|$ variables taking values in $\mathbb{N}_0^s$, yet *not* components of a single variable $\alpha \in \mathbb{N}_0^s$. Letting $\gamma := \sum_{j=1}^{|\beta|} \alpha_j$, the above expression (17) contains a factor $\tilde{\xi}^\gamma \varepsilon^{|\gamma|-|\beta|}$. (Note that since all $|\alpha_j| \geq 1$ we have $|\gamma| \geq |\beta|$.) If $|\gamma| - |\beta| \geq \left[\frac{q+1}{2}\right]$ we are done with that particular term, taking into account that the integral has to be taken over a fixed compact set only. If, on the other hand, $|\gamma| - |\beta| < \left[\frac{q+1}{2}\right]$ and all η_j vanish we may use the assumption on $\tilde{\phi}(\varepsilon, \tilde{x})$ since in this case $|\gamma| < \left[\frac{q+1}{2}\right] + |\beta| \leq q+1$. Finally, if there is at least one η_j nonvanishing then at least one $|\alpha_j| = n+1$, implying $|\gamma| \geq n + |\beta|$. Hence, choosing $n \geq \left[\frac{q+1}{2}\right]$ completes the proof for the case $\alpha = 0$.

To deal with the general case, express the operator ∂_x^α occurring in (16) in terms of operators $\partial_{\tilde{x}}^{\alpha'}$, according to the chain rule. Now apart from certain partial derivatives of components of μ^{-1} (which are bounded on L), Leibniz' rule yields a sum of terms which are similar to those considered above, with certain derivatives $\partial_{\tilde{x}}^{\alpha''}\partial^{\alpha_j}\mu_{i_j}$ and $\partial_{\tilde{x}}^{\alpha'''}\tilde{\phi}$ replacing $\partial^{\alpha_j}\mu_{i_j}$ and $\tilde{\phi}$, respectively. The powers of ξ resp. ε remaining unchanged, the same reasoning as above establishes (16) for arbitrary $\alpha \in \mathbb{N}_0^s$. $\square$

Note that the conclusion of the preceding theorem is also obtained if, instead of $\tilde{\phi} \in \mathcal{C}_b^\infty(I \times \tilde{\Omega}, \mathcal{A}_0(\mathbb{R}^s))$, $\tilde{\phi}$ is only assumed to satisfy the analogs (for $\tilde{\Omega}$) of conditions 1) and 2) of 7.14. Moreover, $\mathcal{A}_0(\mathbb{R}^s)$ can be replaced by $\mathcal{D}(\mathbb{R}^s)$ throughout.

Now **(T7)** and **(T8)** follow from **(T6)** and **(D5)**, due to the particular form of **(D3)** and **(D4)**: Assuming, for example, R to be moderate, R also satisfies condition (Z) of Theorem 10.5. Given $\tilde{K} \subset\subset \tilde{\Omega}$, $\alpha \in \mathbb{N}_0^s$ and $\tilde{\phi} \in \mathcal{C}_b^\infty(I \times \tilde{\Omega}, \mathcal{A}_0(\mathbb{R}^s))$, define ϕ as

in **(T6)**. According to the chain rule,

$$\begin{aligned}\partial_{\tilde{x}}^{\alpha}\Big((\hat{\mu}R)(S_\varepsilon\tilde{\phi}(\varepsilon,\tilde{x}),\tilde{x})\Big) &= \partial_{\tilde{x}}^{\alpha}\left(R(S_\varepsilon\phi(\varepsilon,\mu\tilde{x}),\mu\tilde{x})\right)\\ &= \sum_{\beta:\,|\beta|\leq|\alpha|}\partial_x^{\beta}(R(S_\varepsilon\phi(\varepsilon,x),x))\Big|_{x=\mu\tilde{x}}\cdot g_\beta(\tilde{x})\end{aligned}$$

where each function g_β is a certain sum of products of partial derivatives of components of μ, hence bounded on $\tilde{K}$. R satisfying (Z) of Theorem 10.5, it follows that for some $N\in\mathbb{N}$, $\partial_{\tilde{x}}^{\alpha}\Big((\hat{\mu}R)(S_\varepsilon\tilde{\phi}(\varepsilon,\tilde{x}),\tilde{x})\Big)=O(\varepsilon^{-N})$ uniformly on K. This shows that also $\hat{\mu}R$ is moderate. If, on the other hand, R is assumed to be negligible, R even passes the negligibility test on test objects ϕ being of type $[\mathrm{A}_{\mathrm{l}}^{\infty}]_{K,q}$, according to Theorem 7.9. Now a similar reasoning as in the case of moderateness, this time using Corollary 10.7 in place of Theorem 10.5, establishes the invariance of negligibility under the action of a diffeomorphism. Thus we have shown

7.15 Theorem. (T7) ([28], 25.)
$\mathcal{E}_M$ is invariant under $\hat{\mu}$, i.e., $\hat{\mu}^C$ maps $\mathcal{E}_M(\Omega)$ into $\mathcal{E}_M(\tilde{\Omega})$.

7.16 Theorem. (T8) ([28], 25.)
$\mathcal{N}$ is invariant under $\hat{\mu}$, i.e., $\hat{\mu}^C$ maps $\mathcal{N}(\Omega)$ into $\mathcal{N}(\tilde{\Omega})$.

Having completed all the steps of the general construction scheme in section 3 we finally reach the goal of this section, the definition of the algebra itself:

7.17 Definition. (D6) ([28], 19.)

$$\mathcal{G}(\Omega):=\mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$$

Since the respective ideals of negligible functions are invariant under $D_i:\mathcal{E}_M(\Omega)\to\mathcal{E}_M(\Omega)$ as well as under $\hat{\mu}:\mathcal{E}_M(\Omega)\to\mathcal{E}_M(\tilde{\Omega})$, both these maps factorize via the respective quotients to yield maps (which we denote by the same symbols) $D_i:\mathcal{G}(\Omega)\to\mathcal{G}(\Omega)$ and $\hat{\mu}:\mathcal{G}(\Omega)\to\mathcal{G}(\tilde{\Omega})$. This completes the (functorial) construction of a differential algebra containing $\mathcal{D}'(\Omega)$ in such a way as to extend the usual product on $\mathcal{C}^\infty(\Omega)$.

If we had decided to perform this construction in the J-frame we would have obtained objects $\mathcal{G}^J(\Omega)$ isomorphic to the $\mathcal{G}^C(\Omega)$ above: Indeed, also T^* factorizes via quotients with respect to $\mathcal{N}^J(\Omega)$ resp. $\mathcal{N}^C(\Omega)$, thereby inducing a bijection between the J- and C-variant of the diffeomorphism invariant Colombeau algebra at hand.

Next, we give an example of a test object $\phi(\varepsilon, x)$ in the sense of **(D3)** and a distribution $u \in \mathcal{D}'(\Omega)$ such that on every strip $((0, \varepsilon_0] \times \Omega) \cap D$, the map $x \mapsto (\iota u)(S_\varepsilon \phi(\varepsilon, x), x)$ is not even locally bounded (hence, *a fortiori,* neither smooth) where $\varepsilon_0 \in I$ is arbitrary and

$$D := \{(\varepsilon, x) \mid (\phi(\varepsilon, x), x) \in U_\varepsilon(\Omega)\} = \{(\varepsilon, x) \mid \operatorname{supp} S_\varepsilon \phi(\varepsilon, x) \subseteq \Omega - x\}$$

is the natural maximal domain of definition of $(\iota u)(S_\varepsilon \phi(\varepsilon, x), x)$. This phenomenon is due to the mismatch of the respective smoothness notions for ιu and ϕ. It cannot occur on sets of the form $(0, \varepsilon(K)] \times K$ where $K \subset\subset \Omega$ and $\varepsilon(K)$ is chosen suitably with respect to K, according to the discussion following **(D4)** (Definition 7.3).

7.18 Example. We employ the notation introduced in Example 5.9. In particular, by $x, y; \xi, \eta$ we now denote coordinates of points $(x, y), (\xi, \eta) \in \mathbb{R}^2$, (ξ, η) replacing the former (x, y) due to the need for additional variables x, y in $\phi(x, y)(\xi, \eta)$. Let Ω, ψ, φ, u be defined as in Example 5.9. Choose a smooth non-decreasing function $\nu : \mathbb{R} \to \mathbb{R}$ taking the constant value k on each of the intervals $I_k := [k - \frac{1}{4}, k + \frac{1}{4}]$ $(k \in \mathbb{Z})$, respectively. Define $\phi(\varepsilon, x, y)(\xi, \eta) := \sin \pi y \cdot S_{|\nu(y)|} \psi(\xi, \eta + \nu(y)) \,+\, \varphi(\xi, \eta)$. Then, obviously, $\phi \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$ (note that also $|\nu|$ is smooth). Letting $x := 0$, $y \in I_k \setminus \{k\}$, $\varepsilon := \frac{1}{|k|}$ $(0 \neq k \in \mathbb{Z})$, $(\xi, \eta) \in \Omega$, we obtain

$$\begin{aligned} (\iota u)(S_\varepsilon \phi(\varepsilon, x, y), x, y) &= \sin \pi y \cdot \langle u, S_\varepsilon S_{|\nu(y)|} \psi(\xi, \eta + \nu(y) - y) \rangle \\ &= \sin \pi y \cdot \langle u, \psi(\xi, \eta + k - y) \rangle. \end{aligned}$$

Substituting $t := y - k$ in the last expression (note that $0 < |t| \leq \frac{1}{4}$) yields $(-1)^k \sin \pi t \cdot \langle u, \psi(\xi, \eta - t) \rangle$, the modulus of which tends to infinity as $t \to 0$ (i.e., as $y \to k$) according to Example 5.9.

7.19 Example. We demonstrate that $\bar{\mu}^J$ in general is not τ_0-τ_0-continuous. To this end, it is sufficient to show that $\Phi_\mu : \varphi \mapsto (\varphi \circ \mu^{-1}) \cdot |(\mu^{-1})'| = (\varphi \circ \mu^{-1}) \cdot \frac{1}{|\mu' \circ \mu^{-1}|}$ is not a continuous map from $\mathcal{A}_0(\tilde{\Omega})$ into $\mathcal{A}_0(\Omega)$ with respect to the topology τ induced by the (LF)-topology of $\mathcal{D}(\mathbb{R})$, for some open subsets $\tilde{\Omega}, \Omega$ of $\mathbb{R}$ and $\mu : \tilde{\Omega} \to \Omega$ a suitable diffeomorphism. Consider $\tilde{\Omega} := \Omega := (0, \infty)$; choose $\rho \in \mathcal{D}(\mathbb{R})$ as in Example 5.9, that is, $\operatorname{supp} \rho \subseteq [0, 2]$, $\int \rho = 0$ and $\rho(x) = \exp(-\frac{1}{x})$ for $0 < x \leq 1$. Further, fix any $\psi \in \mathcal{A}_0(\Omega)$. Then $\varphi_n(\xi) := \frac{1}{n} \rho(\xi - \frac{1}{n}) + \psi(\xi)$ defines a sequence converging to ψ in $\mathcal{A}_0(\Omega)$ with respect to τ. Now consider $\mu : \tilde{\Omega} \to \Omega$ defined by $\mu(\xi) := \frac{\xi}{3} \exp(-\frac{3}{\xi})$. Then the sequence formed by $\Phi_\mu(\varphi_n)$ is not even bounded with respect to τ: Evaluating $\Phi_\mu(\frac{1}{n} \rho(\xi - \frac{1}{n}))$ at $\mu(\frac{2}{n})$ yields

$$\frac{1}{n} \cdot \rho\Big(\frac{2}{n} - \frac{1}{n}\Big) \cdot \frac{1}{\mu'(\frac{2}{n})} = \frac{1}{n} \cdot e^{\frac{n}{2}} \cdot \frac{6}{2 + 3n} \to \infty \quad (n \to \infty).$$

To conclude this section, we briefly introduce the notion of association into the diffeomorphism-invariant setting:

7.20 Definition. $R_1,\ R_2 \in \mathcal{G}(\Omega)$ *are called associated* ($R_1 \approx R_2$) *if the following condition holds:* $\forall \psi \in \mathcal{D}(\Omega) \exists q \in \mathbb{N}\ \forall \phi \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_q(\mathbb{R}^s))$*:*

$$\lim_{\varepsilon \to 0} \int (R_1 - R_2)(S_\varepsilon \phi(\varepsilon, x), x)\psi(x)\, dx = 0$$

(where we have used the C-formalism). The concept of associated distribution as well as the basic properties of association are analogous to the non-diffeomorphism invariant case.

8 Sheaf properties

Localization properties of elements of $\mathcal{G}$ are most conveniently formulated in terms of sheaves. This section presents the relevant notations and results.

Let $\Omega \subseteq \Omega'$. Then $U(\Omega') \subseteq U(\Omega)$ and for $R \in \mathcal{G}(\Omega)$ we denote by $R|_{\Omega'}$ the restriction of R to $U(\Omega')$.

8.1 Theorem. $\Omega \to \mathcal{G}(\Omega)$ *is a fine sheaf of differential algebras.*

Proof. Let $\Omega = \bigcup\limits_{\alpha \in A} \Omega_\alpha$. We have to show that

(S1) If $R_1,\ R_2 \in \mathcal{G}(\Omega)$ and $R_1|_{\Omega_\alpha} = R_2|_{\Omega_\alpha}$ for all $\alpha \in A$ then $R_1 = R_2$.

(S2) If for each $\alpha \in A$ we are given $R_\alpha \in \mathcal{G}(\Omega_\alpha)$ such that $R_\alpha|_{\Omega_\alpha \cap \Omega_\beta} = R_\beta|_{\Omega_\alpha \cap \Omega_\beta}$ for all α, β with $\Omega_\alpha \cap \Omega_\beta \neq \emptyset$ then there exists some $R \in \mathcal{G}(\Omega)$ with $R|_{\Omega_\alpha} = R_\alpha$ for all $\alpha \in A$.

(F) If $(\Omega_\alpha)_\alpha$ is locally finite there exists a family of sheaf morphisms $\eta_\alpha : \mathcal{G} \to \mathcal{G}$ such that

  (i) $\sum_{\alpha \in A} \eta_\alpha = \mathrm{id}$.

  (ii) $\eta_\alpha(\mathcal{G}_x) = 0$ for all x in a neighborhood of $\Omega \setminus \Omega_\alpha$ (where $\mathcal{G}_x$ denotes the stalk of $\mathcal{G}$ at x).

Noting that any $K \subset\subset \Omega$ can be written as $K = \bigcup_{\alpha \in A} K_\alpha$, $K_\alpha \subset\subset \Omega_\alpha$, $K_\alpha = \emptyset$ $\forall \alpha \in A \setminus H$, $|H| < \infty$, (S1) follows directly from Definition 7.3. For proving (S2)

we adapt a construction from [28], 21, Theorem 1. Choose a locally finite covering $(W_j)_{j\in\mathbb{N}}$ of Ω such that for each $j \in \mathbb{N}$ there exists $\alpha(j)$ with $\overline{W}_j \subset\subset \Omega_{\alpha(j)}$. Let $(\chi_j)_{j\in\mathbb{N}}$ be a partition of unity subordinate to $(W_j)_{j\in\mathbb{N}}$. Moreover, for each $j \in \mathbb{N}$ let $\theta_j \in \mathcal{D}(\Omega_{\alpha(j)})$, $\theta_j \equiv 1$ in a neighborhood of $\overline{W}_j$ and let $\psi_j \in \mathcal{A}_0(\Omega_{\alpha(j)})$. The map

$$\pi_j(\varphi, x) := (\theta_j(\,.\,+x)\varphi - (\int \theta_j(\xi)\varphi(\xi - x)d\xi - 1)\psi_j(\,.\,+x), x)$$

is smooth from $U(\Omega)$ to $T^{-1}(\mathcal{A}_0(\Omega_{\alpha(j)}) \times \Omega))$ and $\pi_j|_{U(W_j)} = \mathrm{id}$. Then for each $j \in \mathbb{N}$ $R_j : U(\Omega) \to \mathbb{C}$,

$$R_j(\varphi, x) = \begin{cases} \chi_j(x)R_{\alpha(j)}(\pi_j(\varphi, x)) & x \in \Omega_{\alpha(j)} \\ 0 & x \notin \Omega_{\alpha(j)} \end{cases}$$

is smooth and $R_j|_{W_j} = R_{\alpha(j)}|_{W_j}$. Since $(W_j)_{j\in\mathbb{N}}$ is locally finite

$$R(\varphi, x) := \sum_{j\in\mathbb{N}} R_j(\varphi, x)$$

is an element of $\mathcal{E}(\Omega)$. To show that R is moderate we first note that in a neighborhood of any $K \subset\subset \Omega$ only finitely many R_j do not vanish identically, so it is enough to estimate one single R_j. Let $\phi \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$ and choose $L \subset\subset W_j$ with $\mathrm{supp}(\chi_j) \subset\subset L$. There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \le \varepsilon_0$ and all x in a compact neighborhood of L in W_j $\mathrm{supp}(S_\varepsilon\phi(\varepsilon, x)) \subseteq W_j - x$, so $(S_\varepsilon\phi(\varepsilon, x), x) \in U(W_j)$. On this set, $\pi_j = \mathrm{id}$ from which the claim follows by our assumption on R_j.

To establish (S2), by (S1) it suffices to show that $R|_{\Omega_\alpha\cap W_k} = R_{\alpha(k)}|_{\Omega_\alpha\cap W_k}$ for all $k \in \mathbb{N}$ and all $\alpha \in A$ (Note that $R_{\alpha(k)}|_{\Omega_\alpha\cap W_k} = R_\alpha|_{\Omega_\alpha\cap W_k}$ for any α by the assumption in (S2)). Now

$$R|_{\Omega_\alpha\cap W_k} - R_{\alpha(k)}|_{\Omega_\alpha\cap W_k} = \sum_{j\neq k} \chi_j(R_{\alpha(j)} \circ \pi_j - R_{\alpha(k)})|_{\Omega_\alpha\cap W_k} \tag{18}$$

For $K \subset\subset \Omega_\alpha \cap W_k$ and $j \neq k$ set $L = K_1 \cap \mathrm{supp}(\chi_j)$. Let $\phi \in \mathcal{C}_b^\infty(I \times (\Omega_\alpha \cap W_k), \mathcal{A}_0(\mathbb{R}^s))$. Then for x in a neighborhood $M \subset\subset \Omega_\alpha \cap W_j \cap W_k$ of L and sufficiently small ε, $(S_\varepsilon\phi(\varepsilon, x), x) \in U(\Omega_\alpha \cap W_j \cap W_k)$, so $\pi_j(S_\varepsilon\phi(\varepsilon, x), x) = (S_\varepsilon\phi(\varepsilon, x), x)$. Hence the $\mathcal{N}$-estimates for the j-th term in (18) follow from those of $R_{\alpha(j)}|_{\Omega_{\alpha(j)}\cap\Omega_{\alpha(k)}} - R_{\alpha(k)}|_{\Omega_{\alpha(j)}\cap\Omega_{\alpha(k)}}$ on M and the fact that χ_j vanishes identically in a neighborhood of $K \setminus M^\circ$. Finally, for proving (F) set (for $\Omega' \subseteq \Omega$)

$$\eta_\beta|_{\Omega'} := \mathcal{G}(\Omega') \ni R \to \sum_{\{j|\beta=\alpha(j)\}} \chi_j\left(R|_{\Omega'\cap W_j} \circ \pi_j|_{\Omega'}\right)$$

□

9 Separating the basic definition from testing

Having introduced a diffeomorphism invariant Colombeau algebra in section 7, we briefly return to the general discussion of full Colombeau algebras. Regarding the definitions of moderateness resp. negligibility (**(D3)**,**(D4)**), we have adopted the terminology of "testing" in section 3: By Definition **(D1)**, certain "objects" (i.e., functions) R are specified; those which are singled out by Definition **(D3)** as being moderate serve as representatives of elements of the algebra $\mathcal{G} = \mathcal{E}_M / \mathcal{N}$ of generalized functions. The process of deciding whether an object R belongs to $\mathcal{E}_M$ (or to $\mathcal{N}$, respectively) has been called "testing for moderateness resp. negligibility". It is performed by scaling "test objects" of the appropriate type by the operator S_ε (as well as translating them appropriately, whenever the J-formalism is used), plugging them into R and analyzing the resulting behaviour of R on these "paths" as $\varepsilon \to 0$. Depending on the type of Colombeau algebra that is to be constructed, test objects take different forms, for example, φ ([10]), $\phi(\varepsilon)$ ([13]), $\phi(x)$ ([28]) or $\phi(\varepsilon, x)$ ([28], [43]). As opposed to that, the objects R themselves do *not* depend in any way on ε; neither do they depend on x via the first argument (the "φ-slot"). In other words, R accepts only certain pairs (φ, x) as arguments where $\varphi \in \mathcal{A}_0(\mathbb{R}^s)$. $x \in \mathbb{R}^s$. Summarizing, we adopt the following policy:

Defining the objects $R \in \mathcal{E}(\Omega)$ is to be separated strictly from testing them.

This decision is based on the following reasons:

First, it makes the objects simpler and the theory easier to comprehend, yet without restricting its potential. Second, it provides a unifying framework and a common terminology by means of which the different versions of Colombeau algebras and their relations to each other can be analyzed. Finally, it is crucial for the development of algebras of nonlinear generalized functions on smooth manifolds, if this is to be achieved in terms of intrinsic objects; this task is deferred to a subsequent paper ([26], jointly with J. Vickers).

Supposing R to be the image of a non-smooth distribution u unter the corresponding embedding ι into $\mathcal{E}_M$, $R(S_\varepsilon\phi, x)$ can be thought of as a regularization of u: Indeed, $S_\varepsilon\phi$ tends to the delta distribution weakly, due to $\int \phi \equiv 1$. In this sense, $S_\varepsilon\phi$ (e.g., $S_\varepsilon\phi(\varepsilon, x)$) represents a "smoothing process" in its totality, for all $\varepsilon \in I$ and on the whole x-domain Ω. Separating the definition of the objects R from testing them thus amounts to assuming that R does not respond to the smoothing process as a whole but only to its particular stages (represented by single elements φ of $\mathcal{A}_0(\mathbb{R}^s)$).

In the literature, three variants of increasing complexity can be distinguished:

1. The objects R take (certain) pairs (φ, x) as arguments; testing is performed

by inserting $(S_\varepsilon\phi(\varepsilon, x), x)$ into R; the behaviour of $R(S_\varepsilon\phi(\varepsilon, x), x)$ has to be studied.

2. Each object R is given by a family $(R_\varepsilon)_{\varepsilon\in I}$ of functions R_ε as in 1.; testing is performed by investigating $(R_\varepsilon(S_\varepsilon\phi(\varepsilon, x), x))_\varepsilon$.

3. The objects R are defined on some set of pairs $(\mathcal{S}, x)$ where $\mathcal{S} = ((\varepsilon, x) \mapsto S_\varepsilon\phi(\varepsilon, x))$ resp. $\mathcal{S} = ((\varepsilon, x) \mapsto \phi(\varepsilon, x))$ represents some "smoothing process". For testing R, $R(\mathcal{S}, x)$ (which, in turn, has to be dependent on ε!) has to be studied as $\varepsilon \to 0$.

The first of the above variants is the one corresponding to "separation of definitions from testing". Any object from level i gives rise to an object of level $(i+1)$ $(i = 1, 2)$ by the following assignments:

$$\begin{array}{llcll} \text{level } 1 \to \text{level } 2 & R_\varepsilon & := & R & \text{(for all } \varepsilon) \\ \text{level } 2 \to \text{level } 3 & (R(\mathcal{S}, x))(\varepsilon, x) & := & R_\varepsilon(S_\varepsilon\phi(\varepsilon, x), x). & \end{array}$$

Jelínek in [28], Definition 5, definitely chose level 1 for performing his construction: This is made explicit in the last paragraph of item 2 of [28] (see also the discussion in item 3 of [28]). As opposed to that, Definition 5 of [43], e.g., clearly aims at level 3 (the following definition of moderateness (Definition 6), however, is ambiguous since it is not clear in which way $R(\mathcal{S}, x)$ (using our notation) depends on ε). The authors of [13], on the other hand, introduced their basic objects $\mathcal{R} \in \mathcal{E}(\Omega)$ in Definition 2 as smooth maps $\mathcal{R} : \mathcal{A}_0 \times \Omega \to \mathbb{C}^I$ where $\mathcal{A}_0$ denotes a certain set of bounded paths $\phi(\varepsilon)$ (i.e., smoothing processes that are independent of $x \in \Omega$). At first glance, this seems to be a clear indication that it was level 3 they had in mind. In the following line, however, $\mathcal{R}(\phi, x)_\varepsilon$ is specified to be of the form $R(S_\varepsilon\phi(\varepsilon), x)$ (using our notation S_ε for the scaling operator) which has the appearance of level 2, generated by an object R of level 1 in the way described above. As the case may be, using $\mathbb{C}^I$ as range space for $\mathcal{R}$ instead of $\mathbb{C}$ definitely incorporates a certain part of the testing procedure into the definition of the basic objects by introducing ε as parameter from the very beginning.

10 Characterization results

The aim of this section is to derive several characterizations of moderateness and negligibility, respectively, which turned out to be indispensable tools in establishing the diffeomorphism invariance of the algebra constructed in section 7. Moreover,

these characterizations will serve as a basis for an intrinsic formulation of the theory on manifolds ([26]).

We begin by proving Theorem 7.9. To this end, we introduce "descending" sequences of linear projections $P_0, \ldots, P_m$ with the property that P_0 acts as the identity operator on $\mathcal{A}_0(\mathbb{R}^s)$, P_m projects $\mathcal{A}_0(\mathbb{R}^s)$ onto $\mathcal{A}_q(\mathbb{R}^s)$ and the range of P_j is of codimension 1 in the range of P_{j-1} ($j = 1, \ldots, m$).

Fix $q \in \mathbb{N}$ and $r > 0$. Enumerate $\{\beta \mid 1 \leq |\beta| \leq q\}$ in an arbitrary manner as $\{\beta_1, \ldots, \beta_m\}$. Since the family $\{\xi^\beta \mid 0 \leq |\beta| \leq q\}$ is linearly independent in $\mathcal{D}'(B_r(0))$, there exist $\varphi_1, \ldots, \varphi_m \in \mathcal{A}_{00}(\mathbb{R}^s) \cap \mathcal{D}(B_r(0))$ satisfying $\int \xi^{\beta_i} \varphi_j(\xi)\, d\xi = \delta_{ij}$ ($1 \leq i, j \leq m$). Now set

$$P_j := \mathrm{id}_{\mathcal{A}_0(\mathbb{R}^s)} - \sum_{i=1}^{j} \varphi_i \otimes \xi^{\beta_i},$$

that is,

$$P_j(\varphi) := \varphi - \sum_{i=1}^{j} \Big(\int \xi^{\beta_i} \varphi(\xi)\, d\xi \Big) \cdot \varphi_i$$

for $\varphi \in \mathcal{A}_0(\mathbb{R}^s)$, $j = 0, \ldots, m$. Obviously, the operators P_j satisfy the properties mentioned above. As to using projections as defined above and the mean value theorem in the subsequent proof we follow [28].

Proof of Theorem 7.9. It is clear that condition (4^∞) implies the condition specified in Definition 7.3. So let us prove the converse, assuming, in addition, that R is moderate. Let $K \subset\subset \Omega$, $\alpha \in \mathbb{N}_0^s$ and $n \in \mathbb{N}$ be given. In order to derive the required estimates for $\partial^\alpha(R(S_\varepsilon \phi(\varepsilon, x), x))$ (ϕ being a test object of type $[\mathrm{A}_1^\infty]_{K,q}$) we have to provide sets of the form $\mathcal{A}_{0,M_1}(\mathbb{R}^s) \times U \subseteq U_{\varepsilon, M_2}(\Omega)$ ($M_1 \subset\subset \mathbb{R}^s$, $M_2 \subset\subset \Omega$, U an open subset of Ω), the former of which will serve as domain for the operations of calculus to be performed, according to section 6. Choose L with $K \subset\subset L \subset\subset \Omega$ and $r > 0$ such that $\operatorname{supp} \phi(\varepsilon, x) \subseteq B_r(0)$ for $\varepsilon \in I$, $x \in L$. Then $\operatorname{supp} \partial^\beta \phi(\varepsilon, x) \subseteq B_r(0)$ for $\varepsilon \in I$, $x \in K$, $\beta \in \mathbb{N}_0^s$. Let $M_1 := \overline{B}_r(0)$ and pick M_2 satisfying $L \subset\subset M_2 \subset\subset \Omega$. According to Proposition 6.3 there exists $\varepsilon_0 > 0$ such that for $\varepsilon \leq \varepsilon_0$, $U_{\varepsilon, M_2}(\Omega)$ contains $\mathcal{A}_{0,M_1}(\mathbb{R}^s) \times L$. Hence $\mathcal{A}_{0,M_1}(\mathbb{R}^s) \times L^\circ$ is an appropriate domain for our purpose, supposing $\varepsilon \leq \varepsilon_0$ in the sequel.

To obtain a suitable value of q, corresponding to K, α, n fixed above, we proceed as follows:

1. By Theorem 7.12 (note that R was assumed to be moderate), there exists $N \in \mathbb{N}$ such that for every $k = 0, \ldots, |\alpha|$, for every $\beta \in \mathbb{N}_0^s$ with $0 \leq |\beta| \leq |\alpha| + 1$ and for every bounded subset B of $\mathcal{D}(\mathbb{R}^s)$,

$$\partial^\beta \mathrm{d}_1^k (R \circ S^{(\varepsilon)})(\varphi, x)(\psi_1, \ldots, \psi_k) = O(\varepsilon^{-N}) \qquad (\varepsilon \to 0)$$

uniformly for $x \in K$, $\varphi \in B \cap \mathcal{A}_0(\mathbb{R}^s)$, $\psi_1, \dots, \psi_k \in B \cap \mathcal{A}_{00}(\mathbb{R}^s)$.

2. By Definition 7.3 there exists $q \in \mathbb{N}$ such that for every $\phi \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_q(\mathbb{R}^s))$,

$$\partial^\alpha (R(S_\varepsilon \phi(\varepsilon, x), x)) = O(\varepsilon^n) \qquad (\varepsilon \to 0)$$

uniformly for $x \in K$.

3. Without loss of generality, we may suppose $q \geq n + N$.

Now let $\phi \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$ be of type $[\mathrm{A}_1^\infty]_{K,q}$. For the values of q and r determined so far, consider m, $\varphi_1 \dots, \varphi_m$ and the projections $P_0, \dots, P_m$ introduced above. Defining $c_i \in \mathcal{C}_b^\infty(I \times \Omega, \mathbb{C})$ by $c_i(\varepsilon, x) := \int \xi^{\beta_i} \phi(\varepsilon, x)(\xi)\, d\xi$ $(i = 1, \dots, m)$, there exists a constant $C \geq 1$ such that $|\partial^\gamma c_i(\varepsilon, x)| \leq C\varepsilon^q (\leq C)$ for all $x \in K$, $0 \leq |\gamma| \leq |\alpha|$, $i = 1, \dots, m$, due to ϕ being of type $[\mathrm{A}_1^\infty]_{K,q}$. In order to benefit from the moderateness of R in form of the differential condition above, there remains an appropriate bounded subset of $\mathcal{D}(\mathbb{R}^s)$ to be specified. To this end, let

$$B := \Gamma \{\partial^\gamma \phi(\varepsilon, x) \mid x \in K,\ 0 \leq |\gamma| \leq |\alpha|,\ \varepsilon \in I\} + mC \cdot \Gamma \{\varphi_i \mid i = 1 \dots, m\}$$

where ΓA denotes the absolutely convex hull of the subset A of an arbitrary linear space. Since $B \cap \mathcal{A}_0(\mathbb{R}^s) \subseteq \mathcal{A}_{0,M_1}(\mathbb{R}^s)$ and $B \cap \mathcal{A}_{00}(\mathbb{R}^s) \subseteq \mathcal{A}_{00,M_1}(\mathbb{R}^s)$, we may safely use differentials of $R \circ S^{(\varepsilon)}$ for $\varepsilon \leq \varepsilon_0$, evaluated at the respective vectors and $x \in K$.

For $j = 0, \dots, m$, set $\psi_j(\varepsilon, x) := P_j \phi(\varepsilon, x) := \phi(\varepsilon, x) - \sum_{i=1}^j c_i(\varepsilon, x)\varphi_i$. Then, in particular, $\psi_0 = \phi$ and $\psi_m \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_q(\mathbb{R}^s))$. Restricting our attention to $x \in K$, $\varepsilon \leq \varepsilon_0$, the following statements are easily verified:

$$\partial^\gamma \psi_m(\varepsilon, x) \in \mathcal{A}_{q,M_1}(\mathbb{R}^s) \qquad (0 \leq |\gamma| \leq |\alpha|),$$

$$\partial^\gamma \psi_j(\varepsilon, x) \in B,\ \varphi_j \in B \qquad (j = 0, \dots, m,\ 0 \leq |\gamma| \leq |\alpha|),$$

and, for every $t \in [0, 1]$, $j = 1, \dots, m$,

$$\psi_j(\varepsilon, x) + t \cdot c_j(\varepsilon, x)\varphi_j = \phi(\varepsilon, x) - \sum_{i=1}^{j-1} c_i(\varepsilon, x)\varphi_i - (1 - t) \cdot c_j(\varepsilon, x)\varphi_j \in B.$$

By our choice of q, $\sup_{x \in K} |\partial^\alpha (R(S_\varepsilon \psi_m(\varepsilon, x), x))| = O(\varepsilon^n)$ and we are going to show in the sequel that also

$$\sup_{x \in K} |\partial^\alpha (R(S_\varepsilon \phi(\varepsilon, x), x)) - \partial^\alpha (R(S_\varepsilon \psi_m(\varepsilon, x), x))| = O(\varepsilon^n)$$

which will complete the proof of R satisfying (4^∞). For the sake of simplicity, we will omit the argument (ε, x) for the functions ϕ, ψ_j, c_j in the following. According

to the chain rule (the proof for the case $\alpha = 0$ is contained trivially in the argument to follow), we obtain

$$\partial^\alpha(R(S_\varepsilon\phi, x)) - \partial^\alpha(R(S_\varepsilon\psi_m, x)) = \sum_{j=1}^{m} \left[\partial^\alpha(R(S_\varepsilon\psi_{j-1}, x)) - \partial^\alpha(R(S_\varepsilon\psi_j, x))\right] =$$

$$\sum_{j=1}^{m}\sum_{\beta,p} \big[(\partial^\beta \mathrm{d}_1^p(R\circ S^{(\varepsilon)}))(\psi_{j-1}, x)(\partial^{\gamma_1}\psi_{j-1}, \ldots, \partial^{\gamma_p}\psi_{j-1}) - (\partial^\beta \mathrm{d}_1^p(R\circ S^{(\varepsilon)}))(\psi_j, x)(\partial^{\gamma_1}\psi_j, \ldots, \partial^{\gamma_p}\psi_j)\big]$$

where the second sum extends over certain $\beta, p; \gamma_1, \ldots, \gamma_p$. Obviously it is sufficient to derive the desired estimate for each of the terms in square brackets separately; thus fix $\beta, p; \gamma_1, \ldots, \gamma_p$. Substituting $\partial^{\gamma_i}\psi_{j-1} = \partial^{\gamma_i}\psi_j + \partial^{\gamma_i}c_j\varphi_j$ $(i = 1, \ldots, p)$ and using multilinearity and symmetry of the iterated differential transforms the square-bracket term into the sum of

$$(\partial^\beta \mathrm{d}_1^p(R\circ S^{(\varepsilon)}))(\psi_{j-1}, x)(\partial^{\gamma_1}\psi_j, \ldots, \partial^{\gamma_p}\psi_j) - (\partial^\beta \mathrm{d}_1^p(R\circ S^{(\varepsilon)}))(\psi_j, x)(\partial^{\gamma_1}\psi_j, \ldots, \partial^{\gamma_p}\psi_j) \tag{19}$$

and of $2^p - 1$ terms of the form

$$(\partial^\beta \mathrm{d}_1^p(R\circ S^{(\varepsilon)}))(\psi_{j-1}, x)(\partial^{\gamma_{i_1}}c_j\varphi_j, \ldots, \partial^{\gamma_{i_l}}c_j\varphi_j, \partial^{\gamma_{i_{l+1}}}\psi_j, \ldots, \partial^{\gamma_{i_p}}\psi_j) \tag{20}$$

where $\{i_1 \ldots, i_p\} = \{1, \ldots, p\}$ and $1 \le l \le p$. Let us consider (20) first. Observing that $\sup_{x\in K}|\partial^{\gamma_{i_1}}c_j| \le C\varepsilon^q$ and that $\psi_{j-1}, \varphi_j, \partial^{\gamma_{i_2}}c_j\varphi_j, \ldots, \partial^{\gamma_{i_l}}c_j\varphi_j, \partial^{\gamma_{i_{l+1}}}\psi_j, \ldots, \partial^{\gamma_{i_p}}\psi_j$ all are members of B (provided $\varepsilon \le \varepsilon_0$ and $x \in K$) we can make use of the moderateness of R in form of the property derived previously by means of Theorem 7.12 to conclude that

$$\sup_{x\in K}|(\partial^\beta \mathrm{d}_1^p(R\circ S^{(\varepsilon)}))(\psi_{j-1}, x)(\partial^{\gamma_{i_1}}c_j\varphi_j, \ldots, \partial^{\gamma_{i_l}}c_j\varphi_j, \partial^{\gamma_{i_{l+1}}}\psi_j, \ldots, \partial^{\gamma_{i_p}}\psi_j)| = O(\varepsilon^q)\cdot O(\varepsilon^{-N}) = O(\varepsilon^{q-N}) \le O(\varepsilon^n),$$

due to our choice of q. To handle (19), on the other hand, we apply the mean value theorem 4.5 to obtain that the value of the term (19) is contained in the closed convex hull of the set of all (complex) numbers

$$(\partial^\beta \mathrm{d}_1^{p+1}(R\circ S^{(\varepsilon)}))(\psi_j + tc_j\varphi_j, x)(\partial^{\gamma_1}\psi_j, \ldots, \partial^{\gamma_p}\psi_j, c_j\varphi_j) \tag{21}$$

where $0 < t < 1$, $x \in K$. Taking into account that $\sup_{x\in K}|c_j(\varepsilon, x)| \le C\varepsilon^q$ and that each of $\psi_j + tc_j\varphi_j, \partial^{\gamma_1}\psi_j, \ldots, \partial^{\gamma_p}\psi_j, \varphi_j$ is a member of B (provided $\varepsilon \le \varepsilon_0$ and $x \in K$) the

modulus of each of the complex numbers given by (21) can be estimated by $C'\varepsilon^q\varepsilon^{-N}$ for some positive constant C' being independent of t, x, again by the differential condition derived from the moderateness of R. Consequently,

$$\sup_{x\in K} |(\partial^\beta \mathrm{d}_1^p(R\circ S^{(\varepsilon)}))(\psi_{j-1},x)(\partial^{\gamma_1}\psi_j,\dots,\partial^{\gamma_p}\psi_j) - \\ (\partial^\beta \mathrm{d}_1^p(R\circ S^{(\varepsilon)}))(\psi_j,x)(\partial^{\gamma_1}\psi_j,\dots,\partial^{\gamma_p}\psi_j)| = O(\varepsilon^n),$$

thereby concluding the proof. □

The remaining part of this section makes available the means allowing to test moderateness resp. negligibility on test objects ϕ which are defined only on certain subsets of $I\times\Omega$. To this end we first have to provide the technical toolkit for manipulating smooth bounded paths.

10.1 Lemma. (Partition of unity on I) *Let $1 > \varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \ldots \to 0$, $\varepsilon_0 = 2$. Then there exist $\lambda_j \in \mathcal{D}(\mathbb{R})$ ($j = 1,2,\dots$) having the following properties:*

$$1)\ \operatorname{supp}\lambda_j = [\varepsilon_{j+1},\varepsilon_{j-1}] \quad 2)\ \lambda_j(x) > 0 \ \ \textit{for}\ \ x\in(\varepsilon_{j+1},\varepsilon_{j-1})$$

$$3)\ \sum_{j=1}^{\infty}\lambda_j(x)\equiv 1 \ \ \textit{for}\ \ x\in I \quad 4)\ \lambda_j(\varepsilon_j)=1 \quad 5)\ \lambda_1(x)=1 \ \ \textit{for}\ \ x\in[\varepsilon_1,1]$$

Proof. For $j\in\mathbb{N}$, choose $\lambda_j^\circ\in\mathcal{D}(\mathbb{R})$ such that $\operatorname{supp}\lambda_j^\circ = [\varepsilon_{j+1},\varepsilon_{j-1}]$, $\lambda_j^\circ > 0$ on $(\varepsilon_{j+1},\varepsilon_{j-1})$ and $\lambda_0^\circ\in\mathcal{D}(\mathbb{R})$ such that $\operatorname{supp}\lambda_0^\circ = [1,3]$, $\lambda_0^\circ > 0$ on $(1,3)$. Define $\lambda^\circ := \sum\limits_{j=0}^{\infty}\lambda_j^\circ$ and $\lambda_j(x) := \dfrac{\lambda_j^\circ(x)}{\lambda^\circ(x)}$ for $x\in(0,3)$. Then $\sum_{j=1}^\infty\lambda_j(x) = \sum_{j=0}^\infty\lambda_j(x)\equiv 1$ for $x\in I$ and it is easy to see that also the remaining four conditions are satisfied. □

10.2 Lemma. *Assume that for each $K\subset\subset\Omega$ there exists $\varepsilon\in I$ such that the pair (ε,K) satisfies a certain property (P) which is stable with respect to decreasing ε and K in the following sense:*

If (ε_1,K_1) satisfies (P) and $\varepsilon_2\le\varepsilon_1$, $K_2\subseteq K_1$, then also (ε_2,K_2) satisfies (P)
$(0<\varepsilon_1,\varepsilon_2\le 1,\ K_1,K_2\subset\subset\Omega)$.

Then there exists $\psi\in\mathcal{C}^\infty(\Omega)$ satisfying $0<\psi(x)\le 1$ for each $x\in\Omega$ such that an ε as above which is appropriate for K with respect to (P) can be chosen as $\eta_K := \min\limits_{x\in K}\psi(x)$, i.e., (η_K,K) satisfies (P) for each $K\subset\subset\Omega$.

Proof. Let K_n be an increasing sequence of compact subsets of Ω satisfying $K_n \subset K_{n+1}^\circ$ which exhausts Ω ($n \in \mathbb{N}$), e.g. $K_n := \{\lambda \in \Omega \mid |\lambda| \le n \text{ and } \operatorname{dist}(\lambda, \partial\Omega) \ge \frac{1}{n}\}$. Set $K_0 := \emptyset$ and define open sets G_n ($n \in \mathbb{N}$) by $G_1 := \emptyset$, $G_n := K_n^\circ \setminus K_{n-2}$ ($n \ge 2$). Now choose a partition of unity $(\varphi_n)_{n\ge1}$ subordinate to the open covering $(G_n)_{n\ge1}$ of Ω. For $n \ge 2$, let ε_n be such that (ε_n, K_n) satisfies (P), $\varepsilon_n \le \varepsilon_{n-1}$ (we put $\varepsilon_1 := 1$) and define $\psi(x) := \sum\limits_{n=1}^{\infty} \varepsilon_n \varphi_n(x)$. Then it is clear that ψ is smooth and takes its values in I. To complete the proof, let $\emptyset \ne K \subset\subset \Omega$. Let $N \in \mathbb{N}$ be minimal such that K is contained in K_N° and consider $x \in K \setminus K_{N-1}^\circ$: Since $x \in K_N$, it cannot be an element of G_n for $n \ge N+2$. On the other hand, since $x \notin K_{N-1}^\circ$, it neither can belong to G_n for $n \le N-1$ (if $N \ge 2$). Altogether, $x \in G_N \cup G_{N+1}$ which results in $\psi(x) = \varepsilon_N \varphi_N(x) + \varepsilon_{N+1}\varphi_{N+1}(x) \in [\varepsilon_{N+1}, \varepsilon_N]$. Therefore, $\eta_K := \min_{x\in K} \psi(x) \le \min_{x \in K\setminus K_{N-1}^\circ} \psi(x) \le \varepsilon_N$. So finally from $\eta_K \le \varepsilon_N$ and $K \subseteq K_N$ we conclude that (η_K, K) satisfies (P). □

The proof of the preceding lemma also gives the following result (just omit K from the argument and consider $x \in K_N \setminus K_{N-1}^\circ$ instead of $x \in K \setminus K_{N-1}^\circ$):

10.3 Lemma. *Let K_n be an increasing sequence of compact subsets of Ω satisfying $K_n \subset K_{n+1}^\circ$ which exhausts Ω ($n \in \mathbb{N}$) and let $\varepsilon_1 \ge \varepsilon_2 \ge \ldots > 0$ be given. Then there exists $\psi \in \mathcal{C}^\infty(\Omega)$ satisfying $0 < \psi(x) \le \varepsilon_n$ for $x \in K_n \setminus K_{n-1}^\circ$ ($K_0 := \emptyset$).*

10.4 Proposition. (Extension of bounded paths)
Let $\phi : D \to \mathcal{A}_0(\mathbb{R}^s)$ where $D \subseteq I \times \Omega$ such that for each $K \subset\subset \Omega$ there exists $\varepsilon_0 > 0$ and a subset U of D which is open in $I \times \Omega$ having the following properties:

1) *$(0, \varepsilon_0] \times K \subseteq U (\subseteq D)$ and ϕ is smooth on U;*

2) *for all $\alpha \in \mathbb{N}_0^s$, $\{\partial^\alpha \phi(\varepsilon, x) \mid 0 < \varepsilon \le \varepsilon_0,\ x \in K\}$ is bounded in $\mathcal{D}(\mathbb{R}^s)$.*

Then there exist a smooth map $\tilde\phi : I \times \Omega \to \mathcal{A}_0(\mathbb{R}^s)$ and $\sigma \in \mathcal{C}^\infty(\Omega)$ ($0 < \sigma(x) \le 1$ for all $x \in \Omega$) satisfying

1′) *$\tilde\phi = \phi$ on $\{(\varepsilon, x) \in I \times \Omega \mid \varepsilon \le \frac{2}{3}\sigma(x)\}$;*

2′) *$\tilde\phi \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$.*

3′) *$(\tilde\phi(\varepsilon, x), x) \in U_\delta(\Omega)$ for all $(\varepsilon, x) \in I \times \Omega$ and $\delta \le \sigma(x)$.*

In particular, conditions 1′) and 3′) imply that for each $K \subset\subset \Omega$ there exists $\varepsilon_1 := \min\limits_{x\in K} \sigma(x)$ such that $\tilde\phi = \phi$ on an open neighborhood of $(0, \frac{\varepsilon_1}{2}] \times K$ and $(\tilde\phi(\varepsilon, x), x) \in U_\delta(\Omega)$ for all $(\varepsilon, x) \in I \times K$ and $\delta \le \varepsilon_1$.

Proof. First we show that without loss of generality it can be assumed that ε_0 occurring in conditions 1) and 2) above also satisfies the following property 3), in addition to 1) and 2):

3) $(\phi(\varepsilon, x), x) \in U_\delta(\Omega)$ *for all* $0 < \varepsilon, \delta \leq \varepsilon_0$ *and* $x \in K$.

In fact, according to Proposition 6.3 there exists $\eta > 0$ such that $(\phi(\varepsilon, x), x) \in U_\delta(\Omega)$ for all $0 < \varepsilon \leq \varepsilon_0$, $x \in K$ and $0 < \delta \leq \eta$ (observe that $\{\phi(\varepsilon, x) \mid 0 < \varepsilon \leq \varepsilon_0,\ x \in K\}$ is bounded by 2)). Replacing both ε_0 and η by $\min(\varepsilon_0, \eta)$, we see that 1)–3) can be assumed to hold simultaneously.

Now let us say that (ε_0, K) satisfies (P) if 1)–3) are valid for this particular pair (ε_0, K). Then Lemma 10.2 can be applied and provides a function $\sigma \in \mathcal{C}^\infty(\Omega)$ satisfying $0 < \sigma(x) \leq 1$ for each $x \in \Omega$ such that 1)–3) hold for $\min\limits_{x \in K} \sigma(x)$ in place of ε_0. Now let λ_1 be smooth on $\mathbb{R}$, $0 \leq \lambda_1 \leq 1$ and $\lambda_1 \equiv 1$ on $(-\infty, \frac{2}{3}]$, $\lambda_1 \equiv 0$ on $[\frac{5}{6}, +\infty)$. Set $\lambda_2 := 1 - \lambda_1$ and define

$$\tilde{\phi}(\varepsilon, x) := \lambda_1 \left(\frac{\varepsilon}{\sigma(x)} \right) \cdot \phi(\varepsilon, x) + \lambda_2 \left(\frac{\varepsilon}{\sigma(x)} \right) \cdot \phi(\sigma(x), x).$$

$\tilde{\phi}$ is defined on $I \times \Omega$ since the formula above actually involves only values of ϕ on pairs (ε, x) satisfying $\varepsilon \leq \sigma(x)$ and $(0, \sigma(x)] \times \{x\} \subseteq D$ by setting $K := \{x\}$ in 1). In order to show that $\tilde{\phi}$ is smooth we start by observing that ϕ is smooth on some open neighborhood $U (\subseteq D)$ of $D_\sigma := \{(\varepsilon, x) \mid x \in \Omega,\ 0 < \varepsilon \leq \sigma(x)\}$: Setting $K := \{x\}$ in 1) once more yields an open neighborhood U_x of $(0, \sigma(x)] \times \{x\}$ on which ϕ is smooth. It suffices to take $U := \bigcup\limits_{x \in \Omega} U_x$. Now, since $x \mapsto (\sigma(x), x)$ is a smooth map from Ω into U, $\phi(\sigma(x), x)$ is smooth as a function of x. Taking into account that $\operatorname{supp} \lambda_1(\frac{\varepsilon}{\sigma(x)})$ is a subset of $\{(\varepsilon, x) \mid \varepsilon \leq \frac{5}{6}\sigma(x)\}$ and noting that $\sigma(x) > 0$ for all x we see that also the first term in the definition of $\tilde{\phi}$ and, hence, also $\tilde{\phi}$ itself are smooth.

Obviously, $\tilde{\phi}(\varepsilon, x) = \phi(\varepsilon, x)$ for $\varepsilon \leq \frac{2}{3}\sigma(x)$. Thus 1′) is proved.

To show 2′), we have to consider derivatives of $\tilde{\phi}$ with respect to x on sets of the form $I \times K$ where $K \subset\subset \Omega$ is given. Again we set $\varepsilon_1 := \min\limits_{x \in K} \sigma(x)$. First, observe that on $(0, \varepsilon_1] \times K$ all derivatives $\partial^\beta \phi(\varepsilon, x)$ are bounded by 2). Since they are clearly also bounded on the compact set $\{(\varepsilon, x) \mid x \in K,\ \varepsilon_1 \leq \varepsilon \leq \sigma(x)\}$, they are bounded on the whole of $K_\sigma := \{(\varepsilon, x) \mid x \in K,\ 0 \leq \varepsilon \leq \sigma(x)\}$. Now fix $\alpha \in \mathbb{N}_0^s$; to discuss $\partial^\alpha \tilde{\phi}$ in detail, we set

$$\tilde{\phi}_1(\varepsilon, x) := \lambda_1 \left(\frac{\varepsilon}{\sigma(x)} \right) \cdot \phi(\varepsilon, x) \quad \text{resp.} \quad \tilde{\phi}_2(\varepsilon, x) := \lambda_2 \left(\frac{\varepsilon}{\sigma(x)} \right) \cdot \phi(\sigma(x), x).$$

Now

$$\partial^\alpha \tilde{\phi}_1(\varepsilon, x) = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} \partial^\beta \lambda_1 \left(\frac{\varepsilon}{\sigma(x)}\right) \cdot \partial^\gamma \phi(\varepsilon, x).$$

As we have seen above, all derivatives $\partial^\gamma \phi(\varepsilon, x)$ are bounded on K_σ. Expanding $\partial^\beta \lambda_1(\frac{\varepsilon}{\sigma(x)})$ according to the chain rule gives a finite number of terms of the form

$$\lambda_1^{(l)} \left(\frac{\varepsilon}{\sigma(x)}\right) \cdot \varepsilon^l \cdot \partial^{\gamma_1} \left(\frac{1}{\sigma(x)}\right) \cdot \ldots \cdot \partial^{\gamma_l} \left(\frac{1}{\sigma(x)}\right)$$

where $1 \leq l \leq |\beta|$ and $\gamma_1, \ldots, \gamma_l \in \mathbb{N}_0^s$ satisfy $\sum\limits_{i=1}^{l} |\gamma_i| = |\beta|$. Each of these terms is bounded on $I \times K$. Taking into account that all $\partial^\beta \lambda_1(\frac{\varepsilon}{\sigma(x)})$ vanish for $\varepsilon \geq \frac{5}{6}\sigma(x)$ and that K_σ is characterized by $\varepsilon \leq \sigma(x)$, $\partial^\alpha \tilde{\phi}_1$ is bounded on $I \times K$.

The derivatives of $\tilde{\phi}_2$ take the form

$$\partial^\alpha \tilde{\phi}_2(\varepsilon, x) = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} \partial^\beta \lambda_2 \left(\frac{\varepsilon}{\sigma(x)}\right) \cdot \partial^\gamma \phi(\sigma(x), x).$$

The above reasoning showing $\partial^\beta \lambda_1(\frac{\varepsilon}{\sigma(x)})$ to be bounded also applies to $\partial^\beta \lambda_2(\frac{\varepsilon}{\sigma(x)})$. For $\partial^\gamma \phi(\sigma(x), x)$ the chain rule gives a finite sum of terms of the form

$$\partial_\varepsilon^k \partial_x^{\gamma_0} \phi(\sigma(x), x) \cdot \partial_x^{\gamma_1} \sigma(x) \cdot \ldots \cdot \partial_x^{\gamma_k} \sigma(x) \tag{22}$$

where $0 \leq k \leq |\gamma|$ and $\gamma_0, \gamma_1, \ldots, \gamma_k \in \mathbb{N}_0^s$ satisfy $\sum\limits_{i=0}^{k} |\gamma_i| = |\gamma|$. Since $\partial_\varepsilon^k \partial_x^{\gamma_0} \phi$ is bounded on the compact subset $\{(\sigma(x), x) \mid x \in K\}$ of U, all the factors in (22) are bounded on K. Combining this with the boundedness of $\partial^\beta \lambda_2(\frac{\varepsilon}{\sigma(x)})$ shows that also $\partial^\alpha \tilde{\phi}_2$ (and hence $\partial^\alpha \tilde{\phi}$) is bounded on $I \times K$ which completes the proof of $2'$).

Finally, to show $3'$) let $x \in \Omega$, $\delta \leq \sigma(x)$ and conclude from 3) that in particular $(\phi(\sigma(x), x), x) \in U_\delta(\Omega)$. Now for $\varepsilon \leq \sigma(x)$, both $(\phi(\varepsilon, x), x)$ and $(\phi(\sigma(x), x), x)$ belong to $U_\delta(\Omega)$ and so also $(\tilde{\phi}(\varepsilon, x), x)$ does. On the other hand, for $\sigma(x) < \varepsilon \leq 1$ we have $(\tilde{\phi}(\varepsilon, x), x) = (\phi(\sigma(x), x), x) \in U_\delta(\Omega)$. Therefore, for all $\varepsilon \in I$, $(\tilde{\phi}(\varepsilon, x), x) \in U_\delta(\Omega)$.

The last statement of the proof follows from the fact that $\tilde{\phi}$ and ϕ coincide on the open set $\{(\varepsilon, x) \mid \varepsilon < \frac{2}{3}\sigma(x)\}$ which clearly contains $(0, \frac{\varepsilon_1}{2}] \times K$. $\square$

In the following, we will identify each function $\hat{\phi} : I \to \mathcal{C}^\infty(\Omega, \mathcal{A}_0(\mathbb{R}^s))$ in the natural way with the corresponding function $\phi \in \mathcal{C}^{[\infty,\Omega]}(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$ where $\hat{\phi}(\varepsilon)(x) = \phi(\varepsilon, x)$. This identification respects the properties of $\hat{\phi}$ resp. ϕ being smooth (see Theorem 4.2) and/or bounded (in the sense specified in section 2).

10.5 Theorem. (A *to* Z) *The moderateness of an element* R *of* $\mathcal{C}^\infty(U(\Omega))$ *can be tested equivalently on bounded subsets of* $\mathcal{C}^\infty(\Omega, \mathcal{A}_0(\mathbb{R}^s))$*, on arbitrary bounded paths* $\phi : I \to \mathcal{C}^\infty(\Omega, \mathcal{A}_0(\mathbb{R}^s))$ *or on paths of the same form depending smoothly on* ε *(conditions* (A), (B), (C)*, respectively).*

Moreover, equivalent moderateness tests can be performed on larger resp. smaller classes of smooth paths which are distinguished by better resp. poorer properties with respect to the domain of definition of $R(S_\varepsilon\phi(\delta, x), x)$ *(conditions* (D), (E)*;* (Z)*). Formally:*

Let $R \in \mathcal{C}^\infty(U(\Omega))$*. Then the following conditions are equivalent:*

(A) $\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^s\ \exists N \in \mathbb{N}\ \forall \mathcal{B}\,(\textit{bounded}) \subseteq \mathcal{C}^\infty(\Omega, \mathcal{A}_0(\mathbb{R}^s))$
$\exists C > 0\ \exists \eta > 0\ \forall \phi \in \mathcal{B}\ \forall \varepsilon\,(0 < \varepsilon < \eta)\ \forall x \in K$*:*

$$|\partial^\alpha(R(S_\varepsilon\phi(x), x))| \leq C\varepsilon^{-N}$$

(B) $\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^s\ \exists N \in \mathbb{N}\ \forall \phi \in \mathcal{C}_b^{[\infty,\Omega]}(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$
$\exists C > 0\ \exists \eta > 0\ \forall \varepsilon\,(0 < \varepsilon < \eta)\ \forall x \in K$*:*

$$|\partial^\alpha(R(S_\varepsilon\phi(\varepsilon, x), x))| \leq C\varepsilon^{-N}$$

(C) $\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^s\ \exists N \in \mathbb{N}\ \forall \phi \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$
$\exists C > 0\ \exists \eta > 0\ \forall \varepsilon\,(0 < \varepsilon < \eta)\ \forall x \in K$*:*

$$|\partial^\alpha(R(S_\varepsilon\phi(\varepsilon, x), x))| \leq C\varepsilon^{-N}$$

(D) *as condition (C), yet only paths* $\phi \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$ *are considered such that* $(\phi(\varepsilon, x), x) \in U_\delta(\Omega)$ *for all* $\varepsilon, \delta \in I$ *and* $x \in K$*.*

(E) *as condition (C), yet only paths* $\phi \in \mathcal{C}_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^s))$ *are considered such that for each* $L \subset\subset \Omega$ *there exists* δ_0 *having the the property* $(\phi(\varepsilon, x), x) \in U_\delta(\Omega)$ *for all* $(\varepsilon, x) \in I \times L$ *and* $\delta \leq \delta_0$*.*

(Z) $\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^s\ \exists N \in \mathbb{N}\ \forall \phi : D \to \mathcal{A}_0(\mathbb{R}^s))$ *(*D, ϕ *as described below)*
$\exists C > 0\ \exists \eta > 0\ \forall \varepsilon\,(0 < \varepsilon < \eta)\ \forall x \in K$*:* $(\varepsilon, x) \in D$ *and*

$$|\partial^\alpha(R(S_\varepsilon\phi(\varepsilon, x), x))| \leq C\varepsilon^{-N}$$

where $D \subseteq I \times \Omega$ *and for* D, φ *the following holds: For each* $L \subset\subset \Omega$ *there exists* ε_0 *and a subset* U *of* D *which is open in* $I \times \Omega$ *such that*

1) $(0, \varepsilon_0] \times L \subseteq U (\subseteq D)$ *and* ϕ *is smooth on* U*;*

2) for all $\beta \in \mathbb{N}_0^s$*,* $\{\partial^\beta\phi(\varepsilon, x) \mid 0 < \varepsilon \leq \varepsilon_0,\ x \in L\}$ *is bounded in* $\mathcal{D}(\mathbb{R}^s)$*.*

Proof. (A) $\Longrightarrow$ (B) is clear since the image of ϕ, the latter being viewed as a function from I into $\mathcal{C}^\infty(\Omega, \mathcal{A}_0(\mathbb{R}^s))$, forms a bounded subset of $\mathcal{C}^\infty(\Omega, \mathcal{A}_0(\mathbb{R}^s))$.

(B) $\Longrightarrow$ (C) is trivial.

(C) $\Longrightarrow$ (A) Assume to the contrary $\neg$(A), i.e.,

$$\exists K \subset\subset \Omega\ \exists \alpha \in \mathbb{N}_0^s\ \forall N \in \mathbb{N}\ \exists \mathcal{B}\,(\text{bounded}) \subseteq \mathcal{C}^\infty(\Omega, \mathcal{A}_0(\mathbb{R}^s))$$
$$\forall C > 0\ \forall \eta > 0\ \exists \phi \in \mathcal{B}\ \exists \varepsilon\,(0 < \varepsilon < \eta)\ \exists x \in K\colon$$

$$|\partial^\alpha(R(S_\varepsilon \phi(x), x))| > C\varepsilon^{-N}$$

Fix K, α as given by $\neg$(A). (C) yields $N \in \mathbb{N}$ with the property described there ($\forall \phi\,(\mathcal{C}^\infty$, bounded) $\dots$). Now for this N, our assumption $\neg$(A) produces a bounded subset $\mathcal{B}$ of $\mathcal{C}^\infty(\Omega, \mathcal{A}_0(\mathbb{R}^s))$ on which R behaves "badly" in the sense that we can inductively define sequences $\varepsilon_j \in I$, $\phi_j \in \mathcal{B}$ and $x_j \in K$ from which a path $\phi_0(\varepsilon, x)$ can be constructed giving a contradiction to (C) if R is tested on it. Explicitly:

Set $C := 1$, $\eta := 1$ and conclude from $\neg$(A):

$$\exists \varepsilon_1 < 1, \qquad \phi_1 \in \mathcal{B},\ x_1 \in K : \left|\partial^\alpha(R(S_{\varepsilon_1}\phi_1(x), x))\,|_{x=x_1}\right| > 1 \cdot \varepsilon_1^{-N}.$$

Set $C := 2$, $\eta := \min(\frac{1}{2}, \varepsilon_1)$ and conclude from $\neg$(A):

$$\exists \varepsilon_2 < \min(\frac{1}{2}, \varepsilon_1),\ \phi_2 \in \mathcal{B},\ x_2 \in K : \left|\partial^\alpha(R(S_{\varepsilon_2}\phi_2(x), x))\,|_{x=x_2}\right| > 2 \cdot \varepsilon_2^{-N}.$$

Continuing this way, we get $\varepsilon_0 := 1 > \varepsilon_1 > \varepsilon_2 > \ldots \to 0$, $\phi_j \in \mathcal{B}$, $x_j \in K$ satisfying

$$\left|\partial^\alpha(R(S_{\varepsilon_j}\phi_j(x), x))\,|_{x=x_j}\right| > j \cdot \varepsilon_j^{-N}.$$

Take a partition of unity $(\lambda_j)_j$ on I as provided by Lemma 10.1 and define $\phi_0(\varepsilon, x) := \sum_{j=1}^\infty \lambda_j(\varepsilon)\phi_j(x)$ ($\varepsilon \in I$, $x \in \Omega$). Clearly, ϕ_0 is smooth from I into $\mathcal{C}^\infty(\Omega, \mathcal{A}_0(\mathbb{R}^s))$ and its image is bounded since it is contained in the convex hull of $\mathcal{B}$. By construction, we get

$$\left|\partial^\alpha(R(S_{\varepsilon_j}\phi_0(\varepsilon_j, x), x))\,|_{x=x_j}\right| = \left|\partial^\alpha(R(S_{\varepsilon_j}\phi_j(x), x))\,|_{x=x_j}\right| > j \cdot \varepsilon_j^{-N},$$

contradicting $\partial^\alpha(R(S_\varepsilon\phi_0(\varepsilon, x), x)) = O(\varepsilon^{-N})$ as required by (C).

(C) $\Longrightarrow$ (D) is trivial.

(D) $\Longrightarrow$ (E) Let K, α be given and choose N by (D). Consider a path $\phi(\varepsilon, x)$ and a constant $\delta_0 > 0$ appropriate for K, both as specified in (E). Define

$$\tilde{\phi}(\varepsilon, x) := S_{\delta_0}\phi(\delta_0\varepsilon, x) \qquad (\varepsilon \in I,\ x \in \Omega).$$

Since $S_\delta\tilde{\phi}(\varepsilon,x) = S_{\delta\delta_0}\phi(\delta_0\varepsilon,x)$, $(\tilde{\phi}(\varepsilon,x),x) \in U_\delta(\Omega)$ for all $\varepsilon,\delta \in I$ and $x \in K$. (D) now gives $|\partial^\alpha(R(S_\varepsilon\tilde{\phi}(\varepsilon,x),x))| = O(\varepsilon^{-N})$ uniformly on K. However, $\partial^\alpha(R(S_\varepsilon\tilde{\phi}(\varepsilon,x),x)) = \partial^\alpha(R(S_{\delta_0\varepsilon}\phi(\delta_0\varepsilon,x),x))$ which implies

$$|\partial^\alpha(R(S_\varepsilon\phi(\varepsilon,x),x))| = O\left(\left(\frac{\varepsilon}{\delta_0}\right)^{-N}\right) = O(\varepsilon^{-N}).$$

(E) $\Longrightarrow$ (Z) Again let K,α be given; this time, of course, choose N according to (E). By Proposition 10.4, to a given path $\phi(\varepsilon,x)$ as in (Z) there exists a bounded path $\tilde{\phi}(\varepsilon,x)$ satisfying the condition in (E) such that $\tilde{\phi}$ and ϕ coincide on an open neighborhood of $(0,\frac{\varepsilon_1}{2}]\times K$, where ε_1 is some positive constant. Therefore

$$|\partial^\alpha(R(S_\varepsilon\phi(\varepsilon,x),x))| = |\partial^\alpha(R(S_\varepsilon\tilde{\phi}(\varepsilon,x),x))| = O(\varepsilon^{-N}).$$

(Z) $\Longrightarrow$ (C) Setting $D := I\times\Omega$ turns a path in the sense of (C) in a path as required for the application of (Z). □

There is an analog to the preceding theorem giving rise to equivalent definitions of negligibility. Observe, however, that each of the conditions in the following theorem is equivalent to the condition in Definition 7.3 even *without* assuming R to be moderate. This latter property has to be assumed in addition to obtain the definition of negligibility.

10.6 Theorem. (A′ *to* Z′) *In properties* (A)–(Z) *of Theorem 10.5, insert "$\forall n\in\mathbb{N}$" after "$\forall\alpha\in\mathbb{N}_0^s$" and replace "$\exists N\in\mathbb{N}$" by "$\exists q\in\mathbb{N}$", "$\mathcal{A}_0(\mathbb{R}^s)$" by "$\mathcal{A}_q(\mathbb{R}^s)$" and "$C\varepsilon^{-N}$" by "$C\varepsilon^n$", throughout. Then the resulting six conditions* (A′)–(Z′) *are mutually equivalent. If R, in addition, is supposed to be moderate, each of them is equivalent to the negligibility of R.*

The proof of the preceding theorem is analogous to that of Theorem 10.5. Finally, in the proof of **(T8)** in section 7, we have made use of the following variant of (a part of) 10.6:

10.7 Corollary. *In each of conditions* (C′) *and* (Z′) *of Theorem 10.6, replace "$\mathcal{A}_q(\mathbb{R}^s)$" by "$\mathcal{A}_0(\mathbb{R}^s)$" and consider only test objects ϕ all whose derivatives $\partial_x^\alpha\phi$ ($\alpha\in\mathbb{N}_0^s$) have asymptotically vanishing moments of order q on the compact set K at hand. Then the resulting conditions* (C″) *and* (Z″) *are equivalent.*

Proof. (Z″)$\Rightarrow$(C″) being trivial, we have to show the reverse implication. To this end, note that for the property of a test object to have asymptotically vanishing moments on some $K\subset\subset\Omega$, only sets of the form $(0,\varepsilon_0]\times K$ (for some $0<\varepsilon_0\le 1$) are

relevant. Yet it is part of the statement of Proposition 10.4 that the extended path $\tilde{\phi}$ agrees with the given path ϕ on sets of this form, for every given $K \subset\subset \Omega$. Thus the property of having asymptotically vanishing moments is preserved by the extension process $\phi \mapsto \tilde{\phi}$. Now an argument analogous to that used to prove (C) $\Rightarrow$ (Z) in Theorem 10.5 establishes (C$''$) $\Rightarrow$ (Z$''$). □

Conditions (A)–(C) in Theorem 10.5 are due to J. Jelínek ([28], the Remark following Definition 8; the proof of equivalence is only indicated there). The equivalence of condition (Z) with (A)–(C) has to be considered as the technical cornerstone of the diffeomorphism invariance of the Colombeau algebra constructed in section 7. Apart from that, it can be of advantage in certain situations (for example, when dealing with applications) not to have to bother too much about the domains of definition of $R(S_\varepsilon\phi(\delta, x), x)$ being too small. On the other hand, it can be useful to have guaranteed a certain minimum size of these domains; for this reason (D) and (E) have been included in the theorem. Last, but not least we felt the need to give precise meaning to statements like "only $\varepsilon > 0$ small enough is relevant" ([13], p. 361) or "... in the case when the maps Φ_ε [...] are not defined on the same set. We only need that [...] for all $\varepsilon > 0$ sufficiently small." ([28], item 7). In our view, the considerable technical expense required for establishing the results of this section clearly shows the necessity of a rigorous treatment to be given to these matters.

11 Differential Equations

The main application of Colombeau algebras so far has been in the field of differential equations. It is therefore of considerable interest to explore how the changes in the construction of the algebra necessary to ensure diffeomorphism invariance affect the process of solving differential equations in $\mathcal{G}$. To illustrate these changes, in the following we are going to discuss two prototypical examples.

11.1 Example. Consider the initial value problem

$$\begin{aligned} \ddot{x}(t) &= f(x(t))\delta(t) \\ x(-1) &= x_0 \\ \dot{x}(-1) &= \dot{x}_0 \end{aligned} \tag{23}$$

($f : \mathbb{R} \to \mathbb{R}$ smooth) in $\mathcal{G}(\mathbb{R})$. Equations of type (23) arrise, e.g., in geodesic equations in impulsive gravitational waves, cf. [32]. For simplicity, we assume that the initial values x_0 and $\dot{x}_0$ are real numbers.

As in the case of non-diffeomorphism invariant Colombeau theory, solving (23) requires existence and uniqueness results for the classical equation with $\delta(t)$ replaced

by $\varphi(-t)$ (recall that a representative of $\iota(\delta)$ is given by $(\varphi, t) \to \varphi(-t)$). By a standard fixed point argument (cf. [32], Lemma 1) this initial value problem has unique global solutions provided $\mathrm{supp}(\varphi)$ is contained in a sufficiently small neighborhood U (depending on f and the initial conditions) of 0. For $\varphi \in \mathcal{D}(U)$ let $R_1(\varphi, .)$ be this unique solution. Choose some $\chi \in \mathcal{D}(U)$, $\chi \equiv 1$ in a neighborhood V of 0. We claim that $R : (\varphi, t) \to R_1(\chi\varphi, t)$ is a representative of a locally bounded solution to (23) and that this solution is unique. (Note that independence of $\mathrm{cl}[R]$ from χ follows already from the fact that $S_\varepsilon\varphi\chi = S_\varepsilon\varphi$ for any φ and ε sufficiently small). First of all, in order to show that R is smooth on $(U(\mathbb{R}), \tau_2)$ it obviously suffices to establish smoothness of R_1 on that space. To this end, let $s \to (\varphi_s, t_s)$ be a τ_1-smooth curve into some $\mathcal{A}_{0,H}(\mathbb{R}) \times V \subseteq U_N(\mathbb{R})$ as in Theorem 6.6. Then smoothness of solutions of ODEs with respect to a real parameter at once shows that $s \to R_1(\varphi_s, t_s)$ is C^∞, from which the claim follows by definition of smoothness and Theorem 6.6. (We feel that the ease of this kind of argument is a decisive advantage of calculus in convenient vector spaces as used here compared to earlier approaches to differential calculus in locally convex spaces.)

In order to show that $R \in \mathcal{E}_M(\mathbb{R})$, note that for ε sufficiently small

$$R(S_\varepsilon\varphi, t) = x_0 + \dot{x}_0(t+1) + \int_{-1}^{t}\int_{-1}^{s} f(R(S_\varepsilon\varphi, r))S_\varepsilon\varphi(-r)\,dr\,ds\,.$$

The key to proving the desired estimates is the characterization of moderateness given in Theorem 7.12 on the one hand and Remark 6.7 on the other: For φ, ψ varying bounded subsets of $\mathcal{A}_0(\mathbb{R})$ (resp. $\mathcal{A}_{00}(\mathbb{R})$) and ε sufficiently small, iterated differentials of $R \circ S^{(\varepsilon)}$ are well defined and can be calculated according to the usual differentiable structure of $\mathcal{A}_0(\mathbb{R}) \times \mathbb{R}$. In particular, differentiation with respect to φ can be interchanged with integration (see the proof of 4.6), and the chain rule gives, e.g.:

$$d_1(R(S_\varepsilon\varphi, t))[\psi] = \int_{-1}^{t}\int_{-1}^{s} f'(R(S_\varepsilon\varphi, r))d_1(R(S_\varepsilon\varphi, r))[\psi]S_\varepsilon\varphi(-r)\,dr\,ds$$

$$+ \int_{-1}^{t}\int_{-1}^{s} f(R(S_\varepsilon\varphi, r))S_\varepsilon\psi(-r)\,dr\,ds$$

so the result follows (using Gronwall's inequality) by induction. Even in this rather simple example it is quite obvious that to check the moderateness condition 7.2 directly would be extremely tedious (resp. unmanageable for more complicated equations). Moreover, uniqueness can be established similarly without even having to perform any differentiations owing to the remark following Theorem 7.13.

11.2 Example. The semilinear wave equation

$$\begin{aligned}(\partial_t^2 - \Delta)u &= F(u) + H \\ u|_{\{t<0\}} &= 0\end{aligned} \tag{24}$$

with $F \in \mathcal{O}_M(\mathbb{R})$ globally Lipschitz, $F(0) = 0$ and $H \in \mathcal{G}$, $\operatorname{supp} H \subseteq \{t \geq 0\}$ has been treated in the Colombeau framework in [36]. Therefore, we will only indicate those modifications that allow to carry over the existence and uniqueness results achieved there into the current setting. Also, we only treat space dimension 3. Let R_H^η be a representative of H supported in $\{t > -\eta\}$ ($\eta > 0$). For each $\varphi \in \mathcal{A}_0(\mathbb{R}^4)$ let $(x,t) \to R(\varphi, x, t)$ be the smooth solution to

$$\begin{aligned}(\partial_t^2 - \Delta)u(x,t) &= F(u(x,t)) + R_H^\eta(\varphi, x, t) \\ u|_{\{t\leq -\eta\}} &= 0\end{aligned}$$

Then from Kirchhoff's formula we obtain for $t \geq -\eta$

$$R(\varphi, x, t) = \frac{1}{4\pi}\int_{-\eta}^{t} \frac{1}{t-s} \int_{|x-y|=t-s} (F(R(\varphi, y, s)) + R_H^\eta(\varphi, y, s))d\sigma(y)ds \tag{25}$$

Composing R in this formula with a smooth curve as in 11.1, smooth dependence of solutions of Volterra integral equations on real parameters implies smoothness of R in φ and as in the classical case, smooth extension to $t < -\eta$ is possible since $F(0) = 0$. Again using Remark 6.7, φ-differentiation (for $R \circ S^\varepsilon$, ε small, φ, ψ_i (as in Theorem 7.12) varying in bounded sets) interchanges with integration in (25). Hence derivation of the necessary $\mathcal{E}_M$- resp. $\mathcal{N}$-estimates for existence resp. uniqueness of solutions is carried out analogously to the classical case, again due to Theorem 7.12 and the remark following Theorem 7.13.

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